

# On Betti numbers of flag complexes with forbidden induced subgraphs

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## Abstract

We analyze the asymptotic extremal growth rate of the Betti numbers of clique complexes of graphs on  $n$  vertices not containing a fixed forbidden induced subgraph  $H$ .

In particular, we prove a theorem of the alternative: for any  $H$  the growth rate achieves exactly one of five possible exponentials, that is, independent of the field of coefficients, the  $n$ th root of the maximal total Betti number over  $n$ -vertex graphs with no induced copy of  $H$  has a limit, as  $n$  tends to infinity, and, ranging over all  $H$ , exactly five different limits are attained.

For the interesting case where  $H$  is the 4-cycle, the above limit is 1, and we prove a slightly superpolynomial upper bound.

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# 1 Introduction

A central subject of extremal graph theory concerns monotone family of graphs without a fixed subgraph, and its extremal properties – starting with Turán’s theorem, the Erdős-Stone theorem and further generalizations and refinements.

The non-monotone family of graphs  $G$  without fixed *induced* subgraphs have also been the subject of extensive research [CS07]; for structure (e.g., perfect graphs, chordal graphs, coloring, [KKTW01]), enumeration (e.g., [PS92]), as well as extremal properties (e.g., Ramsey theory, Erdős-Hajnal conjecture [EH89, Chu14]).

One of the most far-reaching aspects, however, is also seemingly the most inaccessible. Following Gromov and subsequent work of Davis, Januszkiewicz and Świątkowski, the Betti numbers of clique complexes without small induced cycles are central to the study of nonpositive curvature in certain groups and manifolds. The extremal properties of cohomology under this condition have been prominently studied by Januszkiewicz and Świątkowski [JS03], who created interesting examples of hyperbolic Coxeter groups of very high dimension, something that was previously thought impossible. Here we focus on this fundamental problem from a different perspective, aimed at uniting graph-theoretic and geometric perspectives.

**Question 1.1.** *For any simple finite graph  $H$ , what is the maximal total Betti number over all clique complexes  $\text{cl}(G)$  of graphs  $G$  with at most  $n$  vertices and without an induced copy of  $H$ ?*

Let  $\mathbb{K}$  be any field,  $H$  be any simple finite graph, and

$$b_H(n) = b_H(n, \mathbb{K}) = \max_G \left\{ \sum_{i \geq -1} \dim_{\mathbb{K}} \tilde{H}_i(\text{cl}(G); \mathbb{K}) \right\}$$

where  $G$  runs over all simple graph on at most  $n$  vertices without an induced copy of  $H$ , and  $\tilde{H}_i(\cdot; \mathbb{K})$  denotes the  $i$ th reduced homology with coefficients over  $\mathbb{K}$ . Note that  $b_H(0) = 1$  for any  $H$ , where  $G = \emptyset$  is the only graph in the above

$\max_G$ . We are interested in the growth of  $b_H(n)$  as  $n$  tends to infinity. The results turn out to be, quite interestingly, independent of the coefficient field.

Adamaszek [Ada14] showed that  $b(n) \leq 4^{n/5}$ , for

$$b(n) = \max_G \left\{ \sum_{i \geq -1} \dim_{\mathbb{K}} \tilde{H}_i(\text{cl}(G); \mathbb{K}) \right\}$$

where  $G$  runs over *all* graphs on at most  $n$  vertices. Moreover the maximum is attained by the complete multipartite graph  $K_{5,5,\dots,5}$  when  $n$  is divisible by 5; we deduce that  $\lim_{n \rightarrow \infty} \sqrt[n]{b(n)} = 4^{1/5}$ .

Therefore, if  $H$  is not an induced subgraph of the infinite complete multipartite graph  $K_{5,5,\dots}$ , then  $b_H(n)$  may grow as quickly as  $(4^{1/5})^n$  (and again  $\lim_{n \rightarrow \infty} \sqrt[n]{b_H(n)} = 4^{1/5}$ ).

Thus, it is only interesting to study the function  $b_H(n)$  for induced subgraphs  $H$  of  $K_{5,5,\dots}$ . The (finite) induced subgraphs of  $K_{5,5,\dots}$  are exactly the complete multipartite graphs  $K_{i_1, i_2, \dots, i_m}$  where (without loss of generality)  $5 \geq i_1 \geq \dots \geq i_m \geq 1$ . If  $m = 1$ , then we get the independent set  $I_{i_1}$  on  $i_1$  vertices (it should not be confused with  $K_{i_1} = K_{1,\dots,1}$ , the complete graph on  $i_1$  vertices, which is also an induced subgraph of  $K_{5,5,\dots}$ ).

Adamaszek further showed that for  $H = I_3$ , the growth is exponential but with a smaller base, at most  $\approx 1.2499 < 4^{1/5} \approx 1.3195$ . It is also obvious that, if  $H = K_d$  is a complete graph on  $d$  vertices, then  $\text{cl}(G)$  is at most  $(d-2)$ -dimensional, and thus  $b_{K_d}(n) = O(n^{d-1})$ .

We will prove that the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{b_H(n)}$  exists for any  $H$  and we denote this limit by  $c_H$ . Most strikingly, we will prove a theorem of the alternative:  $c_H$ , depending on  $H$ , can attain one of only 5 different values:

**Theorem 1.2.** *Let  $H$  be any graph. The limit  $c_H = \lim_{n \rightarrow \infty} \sqrt[n]{b_H(n)}$  exists. In addition:*

- (i) *If  $H \not\subseteq K_{5,5,\dots}$ , then  $c_H = 4^{1/5} \approx 1.3195$ .*
- (ii) *For every  $i \in \{1, \dots, 5\}$  there is a value  $c'_i$  with the following property. If  $H =$*

$K_{i_1, \dots, i_m}$  with  $5 \geq i_1 \geq \dots \geq i_m \geq 1$ , then  $c_H = c'_{i_1}$ . Moreover,  $c'_5 = 3^{1/4} \approx 1.3161$ ,  $c'_4 = 2^{1/3} \approx 1.2599$ ,  $c'_3 \in [8^{1/14}, \Gamma_4] \approx [1.1601, 1.2434]$ , and  $c'_2 = c'_1 = 1$ . Here  $\Gamma_4$  is a certain constant which is precisely defined in the Preliminaries.

We summarize our results (including Adamaszek's bounds) in Table 1.

	$H$	$c_H$	lower bound	upper bound
	$H \not\leq K_{5,5,\dots}$	$4^{1/5} \approx 1.3195$	$4^{n/5}$	$4^{n/5}$
$i_1 = 5$	$I_5$ $K_{5,\dots,5}$ ( $m$ parts)	$3^{1/4} \approx 1.3161$	$3^{n/4}$ $(4 \cdot 3^{-5/4})^{m-1} 3^{n/4}$	$3^{n/4}$ $(4 \cdot 3^{-5/4})^{m-1} 3^{n/4}$
$i_1 = 4$	$I_4$	$2^{1/3} \approx 1.2599$	$2^{n/3}$	$2^{n/3}$
$i_1 = 3$	$I_3$	$\in [8^{1/14}, \Gamma_4] \approx [1.1601, 1.2434]$	$8^{n/14}$	$\Gamma_4^n$
$i_1 \leq 2$	$K_{2,2} = C_4$ $K_{2,1,\dots,1}$ ( $m$ parts) $K_{1,1,\dots,1}$ ( $m$ parts)	1	$\Omega(n^{3/2})$ $\Theta(n^{m-1})$ $\Theta(n^{m-1})$	$n^{O(\sqrt{\log n})}$ $\Theta(n^{m-1})$ $\Theta(n^{m-1})$

Table 1: The value  $c_H$  and the upper and lower bounds on  $b_H(n)$  for interesting graphs  $H$ . The lower bounds are valid for infinitely many values of  $n$ .

Now, let us assume that  $H = K_{i_1, \dots, i_m}$  is an induced subgraph of  $K_{5,5,\dots}$  with  $5 \geq i_1 \geq \dots \geq i_m$ . Theorem 1.2 shows that for  $i_1 \in \{3, 4, 5\}$ , the function  $b_H(n)$  grows exponentially. Let  $H \leq G$  denote that  $H$  is an induced subgraph of  $G$ . The following theorem gives more refined bounds for any  $I_5 \leq H \leq K_{5,\dots,5}$ .

**Theorem 1.3.** *If  $H = K_{5,\dots,5}$  is  $m$ -partite,  $m \geq 1$ , then*

$$b_H(n) \leq \left( \frac{4}{3^{5/4}} \right)^{m-1} \cdot 3^{n/4} \approx 1.0131^{m-1} 1.3161^n.$$

*This bound is tight if  $n - 5(m-1)$  is divisible by 4 and positive, and is attained by the  $(m-1) + \frac{n-5(m-1)}{4}$ -fold join consisting of  $m-1$  copies of  $I_5$  and the rest are  $I_4$ .*

The upper bound given in Theorem 1.2 for  $H = I_3$  slightly improves the original bound by Adamaszek, but do not believe it to be optimal yet. We present

it mainly for the proof, which sets up a method how to push Adamaszek's approach further. We believe that by the same method, the obtained value can be further improved, possibly even to the optimal bound, at the cost of a more extensive case analysis.

Regarding  $H = I_4$ , we show that  $c'_4 = c_{I_4} = 2^{1/3}$ . The proof requires an extensive case analysis; therefore, we keep it separately in the appendix. (However, some new ideas are needed as well to perform the analysis.) In fact we show exact bound  $b_{I_4}(n) \leq 2^{n/3}$ ; see Theorem A.2 (in complementary setting, explained in the Preliminaries). This bound is tight if  $n$  is divisible by 3, which is witnessed by the  $n/3$ -fold join of  $I_3$ . In this case, we did not attempt to obtain a more precise bound for  $H = K_{4,\dots,4}$  given the length of the analysis for  $I_4$ .

We now improve the bounds for graphs where the growth is subexponential, specifically, for certain  $H = K_{i_1,\dots,i_m}$  where  $i_j \leq 2$  for any  $j$ .

**Theorem 1.4.** *If  $H = K_{2,2} = C_4$  is the 4-cycle, then there are constants  $c, C > 0$  such that for any  $n$*

$$cn^{3/2} < b_{C_4}(n) < n^{C\sqrt{\log n}}.$$

**Theorem 1.5.** *If  $H = K_{i_1,1,\dots,1}$  where  $i_1 \leq 2$ , then  $b_H(n)$  has a polynomial growth*

$$b_H(n) = \Theta(n^{m-1})$$

where  $m$  is the number of parts in  $H$ .

Note that for  $C_4 = K_{2,2}$  our upper bound on  $b_{C_4}(n)$  is subexponential but superpolynomial. The main problem on the growth of  $b_H(n)$  that remains open is the following.

**Question 1.6.** *For any  $k \geq 2$  and  $l \geq 0$  let  $H = K_{2,\dots,2,1,\dots,1}$  with  $k$  parts of size 2 and  $l$  parts of size 1. Does  $b_H(n)$  have a polynomial growth, namely, is there a function  $f(k, l)$  such that  $b_H(n) < n^{f(k,l)}$  for any large enough  $n$ ?*

A necessary condition for a superpolynomial growth when  $H = C_4$  is that for any positive integer  $d$  there is a graph  $G_d$  with no induces  $C_4$  such that  $\text{cl}(G_d)$  has

a nonvanishing homology in dimension  $> d$ . As mentioned, such constructions exist: Januszkiewicz and Świątkowski [JŚ03] found  $G_d$  such that  $\text{cl}(G_d)$  is a  $d$ -dimensional pseudomanifold, for any positive integer  $d$ .

In Section 2 we overview relevant results of Adamaszek [Ada14], in Section 3 we prove Theorem 1.4, in Section 4 we prove Theorem 1.5, in Section 5 we prove the existence of the limit  $c_H$ , in Section 6 we provide the exponential bounds for  $c_{I_5}, c_{I_3}$  stated in Theorem 1.2 and the refined bounds of Theorem 1.3. Concluding remarks are given in Section 7. Appendix A contains the proof of the optimal bound for  $c_{I_4}$ .

## 2 Preliminaries

We overview some of the results of Adamaszek [Ada14] that will be also useful for us. Following Adamaszek, we present the results in the complementary setting, that is, instead of considering a clique complex over a graph we consider here the independence complex over the complement of the graph. In Section 6 some of the graph theoretical notions that we will meet along the way are more natural in this complementary setting.

Let  $G$  be a graph and we want to estimate the sum of the reduced Betti numbers of the independence complex of  $G$ , denoted by  $\mathbf{b}(G)$  (computed over some fixed field of coefficients). We will occasionally need the following lemma, which easily follows from the Künneth formula.

**Lemma 2.1** ([Ada14, Lemma 2.1(a)]). *Let  $G$  and  $H$  be two graphs. Then  $\mathbf{b}(G \sqcup H) = \mathbf{b}(G)\mathbf{b}(H)$  where  $G \sqcup H$  stands for the disjoint union of  $G$  and  $H$ .*

Given a graph  $G$ , by the symbol  $N[u] = N_G[u]$  we denote the *closed* neighborhood of a vertex  $u$  in  $G$ , that is, the set of neighbors of  $u$  including  $u$ . Given a set  $A$  of vertices of  $G$ , by  $G - A$  we mean the induced subgraph of  $G$  induced by  $V(G) \setminus A$ . We also write  $G - v$  instead of  $G - \{v\}$  for a vertex  $v$  of  $G$ . Let us state another lemma by Adamaszek useful for us.

**Lemma 2.2** ([Ada14, Lemma 2.1(c)]). *For any vertex  $v$  of a graph  $G$  we have  $\mathbf{b}(G) \leq \mathbf{b}(G - v) + \mathbf{b}(G - N[v])$ .*

The lemma follows from the Mayer-Vietoris long exact sequence for the decomposition of a simplicial complex as the union of a star and anti-star of some vertex.

Now, let us assume that  $v$  is a vertex of degree  $d$  of  $G$  and let  $v_1, \dots, v_d$  be all its neighbors (in arbitrarily chosen order). An iterative application of the previous lemma gives the following recurrent bound; see [Ada14, Eq. (5)]. (Note that Adamaszek states the bound in slightly different notation. He also assumes that  $v$  is a vertex of minimum degree. However, this assumption is unimportant in the proof of Eq. (5) in [Ada14]; it is only used in subsequent computations.)

**Lemma 2.3.** *Let  $v$  be a vertex of degree  $d$  and  $v_1, \dots, v_d$  all its neighbors. Then*

$$\mathbf{b}(G) \leq \sum_{i=1}^d \mathbf{b}(G - N[v_i] - \{v_1, \dots, v_{i-1}\}).$$

From this lemma, Adamaszek deduces bounds on  $\mathbf{b}(G)$  for arbitrary graph  $G$  and for a graph  $G$  which is triangle-free. It is very useful for our further approach to describe how to get such bounds from Lemma 2.3.

Given a class  $\mathcal{G}'$  of graphs, let  $\mathbf{b}(\mathcal{G}'; n)$  denote the maximum possible  $\mathbf{b}(G')$  for a graph  $G' \in \mathcal{G}'$  on at most  $n$  vertices, assuming that such a graph exists (otherwise  $\mathbf{b}(\mathcal{G}'; n)$  remains undefined). Let  $\mathcal{G}$  denote the class of all graphs and  $\mathcal{G}_i$  denote the class of the  $K_i$ -free graphs, namely graphs with no copy of the complete graph on  $i$  vertices.

From now on let us assume that  $G$  is a fixed graph with  $n$  vertices. We may also assume that  $G$  does not contain isolated vertices, otherwise  $\mathbf{b}(G) = 0$ . We also set  $n_i$  to be the number of vertices of  $G - N[v_i] - \{v_1, \dots, v_{i-1}\}$  where  $v$  and  $v_1, \dots, v_d$  are as above, for  $i \in [d]$ . In addition, since now we assume that  $v$  is a

$d$	1	2	3	4	5
$\Theta_d$	1	1.2599	1.3161	1.3195	1.3077
$\Gamma_d$	1	1.2207	1.2499	1.2434	1.2293

Table 2: Approximative values of  $\Theta_d = d^{1/(d+1)}$  and  $\Gamma_d$ .

vertex of minimum degree. Lemma 2.3 implies

$$\mathbf{b}(G) \leq d \cdot \mathbf{b}(\mathcal{G}; n - d - 1). \quad (1)$$

if  $G$  is arbitrary graph and

$$\mathbf{b}(G) \leq \sum_{i=0}^{d-1} \mathbf{b}(\mathcal{G}_3; n - i - d). \quad (2)$$

if  $G$  is triangle-free. Indeed, if  $G$  is arbitrary, then  $n_i$  is at most  $n - (d + 1)$  since  $|N[v_{i+1}] \cup \{v_1, \dots, v_i\}| \geq |N[v_{i+1}]| \geq d + 1$ , and if  $G$  is triangle-free, then  $n_i$  is at most  $n - (d + 1 + i - 1)$  since  $N[v_i]$  and  $\{v_1, \dots, v_{i-1}\}$  are in addition disjoint.

In order to conclude a suitable bound on  $\mathbf{b}(G)$ , it is sufficient to plug a suitable function into the formulas above and prove the bound inductively. Concretely, we set  $\Theta_d = d^{1/(d+1)}$  and we set  $\Gamma_d$  to be the unique root on  $[1, 2]$  of the polynomial equation

$$x^{2d} - x^{d-1} - x^{d-2} - \dots - x - 1 = 0.$$

It turns out that the sequence  $\Theta_d$  is increasing on  $[1, 4]$  and decreasing on  $[4, \infty)$ . In particular, it is maximized for  $d = 4$ . Similarly,  $\Gamma_d$  is increasing on  $[1, 3]$  and decreasing on  $[3, \infty]$ , therefore maximized for  $d = 3$ . Later on we will need to know approximative values of  $\Gamma_d$  and  $\Theta_d$  for small  $d$ ; we provide these values for small  $d$  in Table 2.

Now, if we inductively assume that  $\mathbf{b}(\mathcal{G}; k) \leq \Theta_4^k$  for  $k < n$ , then Equation (1) gives

$$\mathbf{b}(G) \leq d \cdot \Theta_4^{n-d-1} = \Theta_4^n \cdot \frac{d}{\Theta_4^{d+1}} \leq \Theta_4^n \cdot \frac{d}{\Theta_d^{d+1}} = \Theta_4^n, \quad (3)$$



which proves  $b(\mathcal{G}, n) \leq \Theta_4^n$ . A similar computation yields  $b(\mathcal{G}_3, n) \leq \Gamma_3^n$ .

The first bound is tight as pointed out by Adamaszek, at least for  $n$  divisible by 5. We will show that the second bound is not tight, and can be improved to  $\Gamma_4^n < \Gamma_3^n$ ; see Section 6.3.

### 3 Subexponential growth for $H = C_4$

We now prove Theorem 1.4. We start with the upper bound:

**Theorem 3.1.**  $b_{C_4}(n) < n^{O(\sqrt{\log n})}$ .

The proof uses the following simple observation: call a nontrivial homology  $d$ -cycle  $z$  in  $\text{cl}(G)$  *d-minimal for  $G$*  if its vertex support, denoted  $z_0$ , satisfies that any strict subset of it is not the vertex support of any nontrivial homology  $d$ -cycle in  $\text{cl}(G)$ . For a subset  $A$  of the vertices of  $G$  let  $G[A]$  denote the induced subgraph on  $A$ , and  $\text{cl}(A) := \text{cl}(G[A])$  for short. Let  $N_G(v)$  denote the set of neighbors of  $v$  in  $G$  (excluding  $v$ ). For a vertex  $v$  in  $z_0$  denote by  $\text{lk}_z(v)$  the  $(d-1)$ -chain in the complex  $\text{cl}(N_z(v))$  induced by the link map, where  $N_z(v)$  is the set  $N_G(v) \cap z_0$ ; clearly  $\text{lk}_z(v)$  is a cycle. Then:

**Lemma 3.2.** *If a cycle  $z$  is  $d$ -minimal for  $G$  and  $v \in z_0$ , then the  $(d-1)$ -cycle  $\text{lk}_z(v)$  is homologically nontrivial in  $\text{cl}(N_z(v))$ .*

*Proof.* Suppose by contradiction there is a  $d$ -chain  $C$  in  $\text{cl}(N_z(v))$  with boundary  $\partial C = \text{lk}_z(v)$ . Then  $\Gamma = z - z_{|\text{star}(v)} + C$  is a  $d$ -cycle whose support is a strict subset of  $z_0$ , where  $z_{|\text{star}(v)}$  is the chain obtained by restricting  $z$  to its summands that support  $v$ . To reach a contradiction to the minimality of  $z$  it is enough to show that  $\Gamma$  is non-trivial. As  $\text{cl}(G)$  is flag, the cycle  $C - z_{|\text{star}(v)}$  is supported on a cone and hence is trivial, so  $C$  and  $z_{|\text{star}(v)}$  are homologous chains and thus  $\Gamma$  and  $z$  are homologous.  $\square$

*Proof of Theorem 3.1.* We show that if  $G$  has no induced  $C_4$  and  $\text{cl}(G)$  has non-trivial homology in dimension  $d$ , then  $G$  must have many vertices, specifically at least  $2^{\binom{d-1}{2}}$  vertices; this implies the claimed bound.

Define the following three functions:

- $\beta(d)$  is the minimal number of vertices in the support of a nontrivial homological  $d$ -cycle  $z$  in  $\text{cl}(G)$ , over all graphs  $G$  as above; we will show that  $\beta(d) \geq 2^{\binom{d-1}{2}}$ .
- $\gamma(d)$  is the minimal number of vertices in the support of a nontrivial homological  $d$ -cycle in  $\text{cl}(G)$  that lie *outside* the star of a vertex  $v$  in the support of the cycle, for all graphs  $G$  as above.
- $\omega(d)$  is the minimum over all graphs  $G$  as above of the maximal size of an independent set in  $G$  that is contained in the support of a nontrivial homological  $d$ -cycle in  $\text{cl}(G)$ .

The value  $\beta(d)$  is clearly realized on graph  $G$  and cycle  $z$  where  $z$  is  $d$ -minimal for  $G$ . By considering the link of a vertex  $v \in z_0$ , Lemma 3.2 immediately gives

$$\beta(d) \geq \beta(d-1) + \gamma(d) + 1.$$

Now if  $u$  and  $w$  are two vertices in the link  $\text{lk}_z(v)$  where  $z$  is  $d$ -minimal for  $G$ , and  $v \in z_0$ , and if  $u, w$  are non-adjacent in  $G$ , then by the no induced  $C_4$  condition, there is no common neighbor of  $u$  and  $w$  outside the star of  $v$  in  $G$ . Further, by looking on the link  $\text{lk}_z(u)$ , the number of vertices outside the star of  $v$  in  $\text{lk}_z(u)$  is, by definition, at least  $\gamma(d-1)$ . Thus, by taking a maximal independent set in  $\text{lk}_z(v)$  we get

$$\gamma(d) \geq \gamma(d-1) \cdot \omega(d-1).$$

First, let us give a simple argument that gives a weaker subexponential upper bound: Clearly  $\omega(d) \geq 2$  and  $\gamma(1) = 2$ , hence  $\gamma(d) \geq 2^d$ , thus  $\beta(d) \geq$

$2^{d+1} + (d - 2)$ . In fact  $\beta(d) \geq 2^{d+1}$  for any  $d$ , so if  $G$  has  $n$  vertices and no induced  $C_4$  then the maximum dimension of a nonzero homology in  $\text{cl}(G)$  is  $\log n$ . Thus, the total Betti number of  $\text{cl}(G)$  is upper bounded by the number of subsets of  $[n]$  of size at most  $1 + \log n$ , and the bound  $b_{C_4}(n) < n^{C \log n}$  follows.

Now we improve the lower bound on  $\omega(d)$ , to achieve the bound claimed in the theorem.

Define  $\omega'(d)$  to be the minimum over all pairs  $(G, a)$  of the size  $|T|$  where: (1)  $G$  has no induced  $C_4$ , (2)  $\text{cl}(G)$  has nontrivial  $d$ -th homology, (3)  $a$  is a vertex in  $G$ , and (4)  $T$  is a maximal size independent set in  $G$  such that  $a \in T$  and  $T \setminus \{a\}$  is contained in some  $d$ -minimal cycle for  $G$ .

As  $\omega(d)$  is realized on a  $d$ -minimal cycle (for appropriate  $G$ ), then  $\omega(d) \geq \omega'(d) - 1$ . Now we aim to show  $\omega'(d) \geq 2\omega'(d - 1) - 2$  for any  $d \geq 2$ . Let us consider a vertex  $a \in G$  such that  $\omega'(d)$  is realized for the pair  $(G, a)$ .

- (1) If  $a$  is not adjacent to any vertex in a  $d$ -minimal cycle of  $G$ , let  $v$  be any vertex of such cycle  $z$ .
- (2) If  $a$  is in some  $d$ -minimal cycle  $z$  of  $G$ , then let  $v$  be any vertex in  $z$  adjacent to  $a$ .
- (3) If  $a$  is not in any  $d$ -minimal cycle of  $G$ , but adjacent to some vertex in such a cycle  $z$  then let  $v$  be such neighbor of  $a$ .

Then, in any case,  $\text{lk}_{\text{cl}(z_0 \cup \{a\})}(v)$  has nontrivial  $(d - 1)$ -st homology following Lemma 3.2. In case (1) and (2), this is trivial. In case (3), note that if  $\text{lk}_{\text{cl}(z_0 \cup \{a\})}(v)$  does not support a homologically nontrivial  $(d - 1)$ -cycle, then  $\text{cl}((z_0 \cup \{a\}) \setminus \{v\})$  supports a homology cycle homologous to  $z$  from which  $a$  cannot be deleted without losing the homology generating cycle, by minimality of  $z$ , similarly as in the proof of Lemma 3.2. Hence  $a$  is in some  $d$ -minimal cycle of  $G$ , contradicting the assumption in (3).

Thus, we can consider a maximal size independent set  $T$  in  $G[N_z(v) \cup \{a\}]$  such that  $a \in T$  and  $T \setminus \{a\}$  is contained in some  $(d - 1)$ -minimal cycle for  $G[N_z(v) \cup \{a\}]$ . By definition,  $|T| \geq \omega'(d - 1)$ . Let  $u$  denote any vertex of  $T \setminus \{a\}$

(it exists as clearly  $|T| \geq 2$ ).

By Lemma 3.2 again,  $\text{lk}_z(u)$  is a nontrivial  $(d-1)$ -cycle in  $\text{cl}(N_z(u))$ . As  $v \in N_z(u)$ , we can consider a maximal size independent set  $T'$  such that  $v \in T'$  and  $T' \setminus \{v\}$  is contained in some  $(d-1)$ -minimal cycle of  $G[N_z(u)]$ . Again,  $|T'| \geq \omega'(d-1)$  by definition.

First, notice that the intersection  $T \cap T'$  is empty, as an element there needs to be both adjacent and nonadjacent to  $u$ . Next, for  $\tilde{T} = T' \cup T \setminus \{u, v\}$  we have  $a \in \tilde{T}$ , and that  $\tilde{T} \setminus \{a\}$  lies in  $z$ . We claim that  $\tilde{T}$  is independent. Indeed, if  $x \in T$  and  $y \in T'$  are in  $\tilde{T}$  and form an edge then  $(y, u, v, x, y)$  is an induced  $C_4$ , a contradiction.

We conclude that

$$\omega'(d) \geq |\tilde{T}| \geq 2\omega'(d-1) - 2.$$

We now show that  $\omega'(2) \geq 3$ . Consider a 2-minimal cycle  $z$  for  $G$  and a vertex  $a \in G$  realizing  $\omega'(2)$ . As  $z$  is nontrivial in  $\text{cl}(G)$ , there exists a vertex  $q \in z_0$  not a neighbor of  $a$  in  $G$ . As  $\text{lk}_z(q)$  is nontrivial in  $\text{cl}(N_z(q))$ , there is a 1-minimal cycle  $z'$  for  $G[N_z(q)]$  supported on a subset of  $\text{lk}_z(q)_0$ ; then  $z'$  is an induced simple cycle of length  $\geq 5$  in  $G$  (length 3 is excluded as  $z'$  is nontrivial, and length 4 is excluded by the no induced  $C_4$  condition). If  $z'$  has length  $> 5$  then it contains an independent set of size 3, say with elements  $s, t, u$ . If  $a$  forms no independent set with any two of them, then  $a$  is connected w.l.o.g. to  $s$  and  $t$ . Then  $(s, a, t, q, s)$  is an induced  $C_4$ , a contradiction. So we assume  $z' = (s, t, u, v, w, s)$ , a  $C_5$ . If  $a$  forms no independent set of size 3 with some two of the vertices of  $z'$ , then  $a$  is a neighbor of some 3 consecutive vertices of  $z'$ , say  $s, t, u$ . Then  $(s, a, u, q, s)$  is an induced  $C_4$ , a contradiction. Thus,  $\omega'(2) \geq 3$ .

We conclude  $\omega(d) \geq \omega'(d) - 1 \geq 2^{d-2}$ , so  $\beta(d) \geq \gamma(d) \geq 2^{\binom{d-1}{2}}$  and hence  $b_{C_4}(n) \leq n^{C\sqrt{\log n}}$ .  $\square$

We now turn to the lower bound. Note that for any prime  $p$ , the graph of the projective plane of order  $p$  is bipartite, connected, with no  $C_4$ , it has  $v/2 = p^2 + p + 1$  vertices on each side and  $e = (p+1)(p^2 + p + 1)$  edges. Thus, its total

Betti number equals the dimension of the first homology group, which equals  $e - v + 1$ . Given  $n$ , add some isolated vertices to the above graph where  $p$  is the largest prime for which  $n/2 \geq p^2 + p + 1$ , to obtain a graph  $G$  with  $n$  vertices. As clearly  $n/2 < (2p)^2 + 2p + 1$ , we conclude that  $\text{cl}(G)$  has total Betti number of order  $\Omega(n^{3/2})$ . Thus:

**Corollary 3.3.** *There exists a constant  $c$  such that for any  $n$  large enough  $b_{C_4}(n) > cn^{3/2}$ .*

Combining Theorem 3.1 and Corollary 3.3 gives Theorem 1.4.

## 4 Polynomial growth for $H \leq K_{2,1,\dots,1}$

We prove Theorem 1.5. Clearly, if  $H = K_d$  is a clique of size  $d$ , then all faces of  $\text{cl}(G)$  have size  $\leq d - 1$  and so  $b_{K_d} = O(n^{d-1})$ . On the other hand, the well known Turán graph on  $n$  vertices and without a  $K_d$ , denoted  $T_{d,n}$ , satisfies that  $\text{cl}(T_{d,n})$  has dimension  $d - 2$  and the  $(d - 2)$ -faces without the least element from each of the  $d - 1$  color classes form a basis of  $H_{d-1}(\text{cl}(T_{d,n}); k)$ ; there are  $\Omega(n^{d-1})$  such faces, thus

$$b_{K_d} = \Theta(n^{d-1}).$$

The other case left to consider is  $H = K_{d+1}^-$ , the complete graph on  $d + 1$  vertices minus one edge. Then, any two simplices of  $\text{cl}(G)$  of dimension  $\geq d - 1$  intersect in a face of dimension at most  $d - 3$ . Thus, an iterated application of the Mayer-Vietoris sequence (or one application of the Mayer-Vietoris spectral sequence) shows that the union of all simplices of dimension  $\geq d - 1$  in  $\text{cl}(G)$  is a complex with vanishing homology in dimensions  $\geq d - 1$ . Thus, the entire complex  $\text{cl}(G)$  has vanishing homology in dimensions  $\geq d - 1$ , and so  $b_{K_{d+1}^-}(n) = O(n^{d-1})$ . As  $K_d \leq K_{d+1}^-$  the lower bound provided by the Turán graph  $T_{d,n}$  applies, and we conclude

$$b_{K_{d+1}^-} = \Theta(n^{d-1}).$$

## 5 Existence of the limit $\lim_{n \rightarrow \infty} \sqrt[n]{b_H(n)}$

From the previous sections, we already know that if  $H = C_4$  or  $H \leq K_{2,1,\dots,1}$  then  $b_H(n)$  has a subexponential growth, and thus  $c_H := \lim_{n \rightarrow \infty} \sqrt[n]{b_H(n)} = 1$  in these cases. Further, Adamaszek's result Theorem 1.2(i) implies that if  $H \not\leq K_{5,5,\dots}$  then  $c_H = \sqrt[5]{4}$ . Our goal now is to show that the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{b_H(n)}$  exists for any  $H$ .

We will work now in the complementary setting. Recall from the Preliminaries that  $\mathbf{b}_H(n) := b_{\overline{H}}(n)$ , where  $\overline{H}$  is the complement of a graph  $H$ .

First, we consider the case when  $H$  is connected.

**Proposition 5.1.** *If  $H$  is connected, then the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{\mathbf{b}_H(n)}$  exists.*

*Proof.* Let, for any positive integer  $n$ ,  $G_n$  be a graph on at most  $n$  vertices which maximizes  $\mathbf{b}_H(n)$ .

For any two positive integers  $m, n$ , the graph  $G_m \sqcup G_n$  does not contain an induced copy of  $H$  as  $H$  is connected and  $G_m$  and  $G_n$  do not contain an induced copy of  $H$ . (We recall that ' $\sqcup$ ' stands for the disjoint union.) Therefore, by Lemma 2.1, we get  $\mathbf{b}_H(m+n) \geq \mathbf{b}_H(m)\mathbf{b}_H(n)$ . By the Fekete lemma for superadditive sequences [Fek23] (see also [vLW01, Lem.11.6]), the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{\mathbf{b}_H(n)}$  exists. In addition, this limit is finite since we already know that  $\mathbf{b}_H(n) \leq \Theta_4^n$  (or we can use the trivial bound  $\mathbf{b}_H(n) \leq 2^n$ ).  $\square$

We now turn to the case where  $H$  is disconnected.

Clearly, for a disjoint union  $H = H_1 \sqcup H_2$ , we have

$$\max_{i=1,2} \liminf_n \sqrt[n]{\mathbf{b}_{H_i}(n)} \leq \liminf_n \sqrt[n]{\mathbf{b}_H(n)}.$$

Denote  $\mathbf{C}_H := \limsup_n \sqrt[n]{\mathbf{b}_H(n)}$ . We now show the reverse inequality, which then implies, in the original setting, that for  $H = H_1 * H_2$ , if the limits

$$c_{H_i} = \lim_{n \rightarrow \infty} \sqrt[n]{b_{H_i}(n)}$$

exist then the limit of  $\sqrt[n]{b_H(n)}$  exists and equals  $c_H = \max(c_{H_1}, c_{H_2})$ .

**Proposition 5.2.** *Let  $H_1$  and  $H_2$  be any two graphs. Then  $C_{H_1 \sqcup H_2} = \max(C_{H_1}, C_{H_2})$ .*

*Proof.* Obviously,  $\max(C_{H_1}, C_{H_2}) \leq C_{H_1 \sqcup H_2}$ , since  $H_1$  and  $H_2$  are induced subgraphs of  $H_1 \sqcup H_2$ . Therefore, it is sufficient to prove  $C_{H_1 \sqcup H_2} \leq \max(C_{H_1}, C_{H_2})$ . For simplicity of subsequent formulas, let  $\alpha = C_{H_1 \sqcup H_2}$  and  $\alpha_i := C_{H_i}$  for  $i \in \{1, 2\}$ . Without loss of generality, we will assume that  $\alpha_1 \geq \alpha_2$ , that is, our task is to show that  $\alpha \leq \alpha_1$ . We will achieve this task by showing that  $\alpha \leq \alpha_1 + \varepsilon$  for any  $\varepsilon > 0$ .

Form now on, let us fix  $\varepsilon > 0$ . We also fix a large enough integer parameter  $p$  which depends on  $\varepsilon$ , but we will describe the exact dependency later on. Now let  $G_n$  be a graph on at most  $n$  vertices which maximizes  $b_{H_1 \sqcup H_2}(n)$ , in particular, it does not contain an induced copy of  $H_1 \sqcup H_2$ . By the definition of  $\alpha_1$  we get

$$b_{H_1}(n) \leq k(\varepsilon)(\alpha_1 + \varepsilon)^n \quad (4)$$

for every  $n$  where  $k(\varepsilon)$  is a large enough constant depending only on  $\varepsilon$ . Since  $\alpha_2 \leq \alpha_1$ , we can also assume that

$$b_{H_2}(n) \leq k(\varepsilon)(\alpha_1 + \varepsilon)^n \quad (5)$$

eventually by adjusting  $k(\varepsilon)$ .

Our aim is to show by induction that

$$b(G_n) \leq 2^p k(\varepsilon)(\alpha_1 + \varepsilon)^n. \quad (6)$$

Note that this inequality is true for  $n = 1$  since  $\alpha_1 \geq 1$ .

It remains to prove Eq. (6) for a fixed  $n$  assuming that it is true for every smaller value. Let us distinguish several cases.

In the first case we assume that  $G_n$  does not contain an induced copy of  $H_2$ . Then we get the desired inequality directly from Eq. (5).

In the second case, let us assume that there are at most  $p$  vertices of  $G_n$  such that when we remove these vertices, we get a graph which does not contain an induced copy of  $H_1$ . Our next task is to show that in this case,  $\mathbf{b}(G_n) \leq 2^p k(\varepsilon)(\alpha_1 + \varepsilon)^n$  which implies desired Eq. (6). Let  $u_1, \dots, u_j$  be the removed vertices from  $G$  and let  $G'$  be the resulting graph. Let  $G''$  be any induced subgraph of  $G$  and let  $i := |V(G'') \cap \{u_1, \dots, u_j\}|$  be the number of vertices  $u_1, \dots, u_j$  in  $G''$ . We will prove by induction in  $i$  that

$$\mathbf{b}(G'') \leq 2^i k(\varepsilon)(\alpha_1 + \varepsilon)^n. \quad (7)$$

When we specify Eq. (7) to  $G$ , that is,  $i = p$ , we get the desired inequality.

The first induction step for  $i = 0$  follows from the fact that  $G''$  is  $H_1$ -free in this case and from Eq. (4). (We could get a better bound since the number of vertices of  $G''$  is (typically) less than  $n$ , but we do not need such an improvement.)

The second induction step for  $i > 0$  follows directly from Lemma 2.2 by removing one of the vertices  $u_1, \dots, u_j$  which is also a vertex of  $G''$ .

Finally, we distinguish a third case when we assume that  $G_n$  contains an induced copy of  $H_2$  and after removing any  $p$  vertices from  $G_n$  we still get a graph that contains an induced copy of  $H_1$ . Let  $H'_2$  be an induced copy of  $H_2$  in  $G$  and let  $h_i$  be the number of vertices of  $H_i$  for  $i \in \{1, 2\}$ . We will prove that  $H'_2$  contains a vertex of degree at least  $\frac{p-h_2}{h_2}$  in  $G_n$ . It is sufficient to show that there are more than  $p - h_2$  edges connecting  $H'_2$  and the remainder of  $G$ . For contradiction, there are at most  $p - h_2$  such edges. Let  $H'$  be the induced subgraph of  $G$  consisting of  $H_2$  and all neighbors of vertices of  $H_2$  inside  $G$ . Then  $H'$  has at most  $p$  vertices. Consequently, there is an induced copy  $H'_1$  of  $H_1$  inside the induced subgraph of  $G_n$  obtained from  $G_n$  by removing the vertices of  $H'$  by our assumption of this distinguished case. By the definition of  $H'$ , the two copies  $H'_1$  and  $H'_2$  are connected by no edge and therefore we have found an induced copy of  $H_1 \sqcup H_2$ , this is a contradiction.



$H'_2$  contains a vertex  $v$  of degree  $d \geq \frac{p-h_2}{h_2}$ . Lemma 2.2 gives

$$\mathbf{b}(G_n) \leq \mathbf{b}(G_n - v) + \mathbf{b}(G_n - N_{G_n}[v]).$$

Note that  $G_n - v$  has at most  $n - 1$  vertices,  $G_n - N_{G_n}[v]$  has at most  $n - d - 1$  vertices and both these graphs do not contain an induced copy of  $H_1 \sqcup H_1$  since they are induced subgraphs of  $G_n$ . Therefore, the induction in  $n$  gives us

$$\mathbf{b}(G_n) \leq 2^p k(\varepsilon) \left( (\alpha_1 + \varepsilon)^{n-1} + (\alpha_1 + \varepsilon)^{n-d-1} \right).$$

It is easy to check that

$$(\alpha_1 + \varepsilon)^{-1} + (\alpha_1 + \varepsilon)^{-d-1} \leq 1$$

if  $d$  is large enough, that is, if  $p$  is large enough, for fixed  $\varepsilon$ , since  $\alpha_1 \geq 1$ . Combining the two above-mentioned inequalities, we get the desired inequality (6).  $\square$

Combining Propositions 5.1 and 5.2 we conclude the existence of the limit  $c_H$  promised in Theorem 1.2, and that for  $H = K_{i_1, \dots, i_m}$  as in Theorem 1.2(ii) indeed  $c_H = c_{I_{i_1}}$ .

To finish the proof of Theorem 1.2(ii) we need to find or bound the constants  $c_{I_k}$  for  $k = 3, 4, 5$  (clearly  $c_{I_1} = 1 = c_{I_2}$ , realized by the empty graph). This we do in the next Section 6.

## 6 Comparing the exponential growth for graphs

$$I_3 \leq H \leq K_{5,5,\dots}$$

In this section we also work in complementary setting, as described in the Preliminaries.

## 6.1 $K_5$ -free graphs

Recall that  $\mathcal{G}_5$  denotes the class of  $K_5$ -free graphs, namely, it consists of the graphs with no induced  $K_5$ . In this case, the upper bound on the homology growth can be improved from  $\Theta_4^n$  to  $\Theta_3^n$ , which is tight.

**Proposition 6.1.**  $b(\mathcal{G}_5; n) \leq \Theta_3^n$ .

*Proof.* Let  $G$  be a  $K_5$ -free graph with  $n$  vertices. The proof is by induction on  $n$ . The base case  $n = 0$  trivially holds as  $b(\emptyset) = 1$ . We may also assume that  $G$  does not contain an isolated vertex otherwise  $b(G) = 0$ .

Let  $d$  be the minimum degree of  $G$ . If  $d \neq 4$ , then the same computation as in Equation (3) gives (note that  $\Theta_3 \geq \Theta_d$  if  $d \neq 4$ ; see Table 2):

$$b(G) \leq \sum_{i=0}^{d-1} \Theta_3^{n-d-1} = \Theta_3^n \frac{d}{\Theta_3^{d+1}} \leq \Theta_3^n \frac{d}{\Theta_d^{d+1}} = \Theta_3^n.$$

It remains to consider the case  $d = 4$ . That is,  $v$  has neighbors  $v_1, \dots, v_4$ . As  $G$  is  $K_5$  free, there is at least one missing edge among these neighbors. For simplicity, we can assume that this missing edge is  $v_1v_2$  since we can choose the order of the neighbors of  $v$ . We deduce that  $n_i \leq n - d - 1 = n - 5$  for  $i \in \{1, 2, 3, 4\}$  as usual, where  $n_i$  is the number of vertices of  $G - N[v_i] - \{v_1, \dots, v_{i-1}\}$ . However, in addition, we can deduce that  $n_2 \leq n - 6$  because  $N[v_2]$  does not contain  $v_1$ . Therefore, Lemma 2.3 gives (using that  $f(n) = \Theta^n$  is increasing)

$$b(G) \leq 3\Theta_3^{n-5} + \Theta_3^{n-6} = \Theta_3^n \Theta_3^{-6} (3\Theta_3 + 1).$$

Now, the equation  $3x + 1 = x^6$  has a root  $x_0 \doteq 1.3038 < \Theta_3$  (this is the only root on  $[1, 2]$ ), and therefore it is easy to deduce that  $3\Theta_3 + 1 \leq \Theta_3^6$  as  $\Theta_3 \geq x_0$  (one can also put directly  $3\Theta_3 + 1$  and  $\Theta_3^6$  into a calculator). This gives the desired bound  $b(G) \leq \Theta_3^n$ .  $\square$

The bound provided by Proposition 6.1 is tight for  $n$  divisible by 4, as the disjoint union of  $n/4$  copies of  $K_4$  shows. For  $n$  not divisible by 4 change the sizes

of one or two components such that each of them have size  $> 1$ , to conclude  $\mathbf{b}(\mathcal{G}_5; n) \geq \frac{2}{9}\Theta_3^n$ . Thus, we get the following corollary.

**Corollary 6.2.**  $c_{I_5}(n) = \Theta_3$ .

## 6.2 $mK_5$ -free graphs

Let  $mK_5$  denote the disjoint union of  $m$  copies of  $K_5$ . By Theorem 1.2(ii) and Proposition 6.1 we already know that for any  $I_5 \leq H \leq mK_5$ ,  $c_H = \Theta_3$ . Here we refine the upper bound on  $\mathbf{b}_{mK_5}(n)$ , as asserted in Theorem 1.3.

**Proposition 6.3.** *Let  $G$  be an  $mK_5$ -free graph (meaning that  $mK_5$  does not appear as an induced subgraph of  $G$ ) with  $n$  vertices. Then*

$$\mathbf{b}(G) \leq 4^{m-1}\Theta_3^{n-5(m-1)} = \frac{\Theta_4^{5(m-1)}}{\Theta_3^{5(m-1)}}\Theta_3^n \doteq 1.0131^{m-1} \cdot 1.3161^n.$$

*Proof.* We prove the result by a double induction. The outer induction is in  $m$ , the inner induction is in  $n$ . The case  $m = 1$  was proved in the previous section, thus we can assume  $m \geq 2$ .

First, let us assume that  $G$  contains  $k$  isolated copies of  $K_5$  for some  $k > 0$ . Let  $G'$  be  $G$  without these copies. Note that  $k \leq m - 1$  and that  $G'$  is  $(m - k)K_5$ -free. Then

$$\mathbf{b}(G) = \mathbf{b}(G')\mathbf{b}(K_5)^k \leq \left(4^{m-k-1}\Theta_3^{n-5k-5(m-k-1)}\right) \cdot 4^k$$

where the equality follows from Lemma 2.1 and the inequality follows from the induction and from  $\mathbf{b}(K_5) = 4$ . That is,  $\mathbf{b}(G) \leq 4^{m-1}\Theta_3^{n-5(m-1)}$  as desired.

If  $G$  does not contain an isolated copy of  $K_5$  then we proceed analogously as in the previous subsection. We let  $d$  be the minimum degree of  $G$  and we consider a vertex  $v$  of degree  $d$  and its neighbors.

If  $d \neq 4$ , then Lemma 2.3 implies

$$\begin{aligned} \mathbf{b}(G) &\leq d \cdot 4^{m-1} \Theta_3^{n-d-1-5(m-1)} = 4^{m-1} \Theta_3^{n-5(m-1)} \frac{d}{\Theta_3^{d+1}} \\ &\leq 4^{m-1} \Theta_3^{n-5(m-1)} \frac{d}{\Theta_d^{d+1}} = 4^{m-1} \Theta_3^{n-5(m-1)}. \end{aligned}$$

Now, let us assume that  $d = 4$ . Since we assume that  $G$  has no isolated  $K_5$ , we either miss some edge among the neighbors  $v_1, \dots, v_4$ , or the degree of some of the vertices  $v_1, \dots, v_4$  is greater than 4. In both cases, Lemma 2.3 provides us with a bound

$$\begin{aligned} \mathbf{b}(G) &\leq 3 \cdot 4^{m-1} \Theta_3^{n-5-5(m-1)} + 4^{m-1} \Theta_3^{n-6-5(m-1)} = 4^{m-1} \Theta_3^{n-5(m-1)} \Theta_3^{-6} (3\Theta_3 + 1) \\ &\leq 4^{m-1} \Theta_3^{n-5(m-1)} \end{aligned}$$

as wanted. (Here we again use the inequality  $3\Theta_3 + 1 \leq \Theta_3^6$  explained at the end of the proof of Proposition 6.1.)  $\square$

### 6.3 $K_3$ -free graphs

We recall that it was explained in the Preliminaries how to get Adamaszek's bound  $c_{I_3} \leq \Gamma_3$ . We aim to get an improved bound  $c_{I_3} \leq \Gamma_4$ . The idea behind the improvement is that a more detailed combinatorial analysis of  $N[v_i]$ , in the setting of Lemma 2.3, reveals one of the following three options. Either  $d \neq 3$  and we can use a bound with  $\Gamma_4$ , or  $d = 3$  and  $v$  can be chosen so that some of the neighbors of  $v$  has degree at least 4 which again improves the bound, or, finally (assuming connectedness),  $G$  is a cubic graph which means that  $N[v_i]$  is a 2-degenerate graph, which again yields improving the bound.

Before addressing general triangle free graphs, it is useful first to give an upper bound on  $\mathbf{b}(G)$  for triangle-free graphs  $G$  which are 2-degenerate.

### 6.3.1 2-degenerate triangle free graphs

Let  $\mathcal{D}_k$  be the class of  $k$ -degenerate graphs, that is graphs, such that for every  $G$  in  $\mathcal{D}_k$  and for every (induced) subgraph  $G'$  of  $G$ , the minimum degree of  $G'$  is at most  $k$ .

**Proposition 6.4.** *Let  $G \in \mathcal{D}_2$  be a triangle-free graph on  $n$  vertices. Then*

$$\sqrt[n]{\mathbf{b}(G)} \leq \Gamma_2 \doteq 1.2207.$$

The bound  $\Gamma_2$  is very probably not an optimal one in this case. However, it is sufficient for our purposes.

*Proof.* The proof is essentially the same as the proof of Adamaszek's bound for triangle free graphs using, in addition, the fact that the minimum degree is at most 2. Assume  $G$  has no isolated vertex, else the assertion is trivial, as  $\mathbf{b}(G) = 0$  in this case.

Let  $v$  be a vertex of minimum degree  $d$  and  $v_1$  and  $v_2$  (or just  $v_1$ ) be its neighbors. If  $d = 2$ , Lemma 2.3 yields

$$\mathbf{b}(G) \leq \Gamma_2^{n-3} + \Gamma_2^{n-4} = \Gamma_2^n \Gamma_2^{-4} (\Gamma_2 + 1) = \Gamma_2^n.$$

In the induction, we crucially use that the subgraphs  $G - N[v_1]$  and  $G - N[v_2] - v_1$  are also triangle-free graphs in  $\mathcal{D}_2$ .

If  $d = 1$ , we even get  $\mathbf{b}(G) \leq \Gamma_2^{n-2} < \Gamma_2^n$  from Lemma 2.3. □

### 6.3.2 General triangle-free graphs

Here we prove the promised bound, namely,

**Proposition 6.5.** *Let  $G$  be a triangle-free graph on  $n$  vertices. Then*

$$\mathbf{b}(G) \leq \Gamma_4^n.$$

*Proof.* As usual, the proof is by induction on  $n$ , again  $d$  is the minimum degree,  $v$  is a vertex of the minimum degree and  $v_1, \dots, v_d$  are its neighbors.

First, we can assume that  $G$  is connected. Indeed, if  $C_1, \dots, C_k$  are the components of  $G$  then we can deduce  $\mathbf{b}(G) \leq \Gamma_4^n$  from  $\mathbf{b}(G) = \mathbf{b}(C_1) \cdots \mathbf{b}(C_k)$  (see Lemma 2.1) and from the induction.

If  $d \neq 3$ , then we deduce

$$\mathbf{b}(G) \leq \Gamma_4^n.$$

from induction analogously to the computations in the proof of Proposition 6.1. Indeed

$$\mathbf{b}(G) \leq \sum_{i=0}^{d-1} \Gamma_4^{n-i-d-1} = \Gamma_4^n \sum_{i=0}^{d-1} \Gamma_4^{-i-d-1} \leq \Gamma_4^n \sum_{i=0}^{d-1} \Gamma_d^{-i-d-1} = \Gamma_4^n.$$

The first inequality follows from the induction analogously to Eq. (2). The last equality follows from the definition of  $\Gamma_d$ . Also note that  $\Gamma_4$  is the largest value among  $\Gamma_d$ , with  $d \neq 3$ .

It remains to consider the case  $d = 3$ . We will distinguish two subcases.

In the first subcase,  $G$  is not a cubic graph (3-regular). That means, it contains two vertices, one of them of degree 3 and the second one of degree greater than 3. Thus we can adjust our choice of  $v$  and its neighbors  $v_1, v_2, v_3$  so that the degree of  $v_1$  is at least 4. This means that  $n_1 \leq n - 5$ ,  $n_2 \leq n - 5$ , and  $n_3 \leq n - 6$ , as there is no edge between  $v_1, v_2, v_3$ . Lemma 2.3 now gives a bound

$$\mathbf{b}(G) \leq \Gamma_4^n (\Gamma_4^{-5} + \Gamma_4^{-5} + \Gamma_4^{-6}) = \Gamma_4^n \Gamma_4^{-6} (2\Gamma_4 + 1).$$

The equation  $2x + 1 = x^6$  has a unique solution  $x_1 \doteq 1.2298$  on  $[1, 2]$  and we can deduce that  $\mathbf{b}(G) \leq \Gamma_4^n$  since  $\Gamma_4 \geq x_1$ ; see Table 2.

In the second subcase we assume that  $G$  is a (connected) cubic graph. In this subcase, we will not save the value on the exponents, but we will save it on the bases. More concretely, in this case we crucially use that the graphs  $G_i - N[v_i] - \{v_1, \dots, v_{i-1}\}$  belong to  $\mathcal{D}_2$  since they are proper subgraphs of a

connected cubic graph. Therefore, we can use Proposition 6.4 and together with Lemma 2.3, for  $n \geq 7$ , we deduce

$$\mathbf{b}(G) \leq \Gamma_2^n (\Gamma_2^{-4} + \Gamma_2^{-5} + \Gamma_2^{-6}) \leq \Gamma_4^n.$$

It is easy to check that for  $n \leq 6$ , the only possible cubic triangle-free graph is  $K_{3,3}$ . In this case  $\mathbf{b}(K_{3,3}) = 1$  and the required inequality is satisfied as well.  $\square$

## 7 Concluding remarks

As mentioned in the Introduction, we still do not know whether there exists a graph  $H$  for which  $b_H(n)$  grows subexponentially and superpolynomially. See Question 1.6 for the candidates for such  $H$ .

The computation of  $b_H(n)$  reduces to graphs with *exactly*  $n$  vertices:

*Monotonicity.* By definition, for any graph  $H$  clearly  $b_H(n)$  is weakly increasing. Let  $b_H^-(n)$  be the maximum total Betti number among all graphs with no induced copy of  $H$  and with *exactly*  $n$  vertices. In fact,

**Observation 7.1.** *For any graph  $H$ , the function  $b_H^-(n)$  is weakly increasing in  $n$ .*

*Proof.* First note that when adding to  $G$  an isolated vertex  $v$ , the total Betti number of  $\text{cl}(G \sqcup v)$  is one more than of  $\text{cl}(G)$ , where the 0th Betti number is increased by one. Thus, the result holds for  $H$  with no isolated vertex. Next, for  $H = H' \sqcup u$ ,  $G$  a maximizer of  $b_H^-(n)$  and a vertex  $w \in G$ , let  $G'$  be obtained from  $G$  by adding a new vertex  $v$  and connecting it to  $w$  and all neighbors of  $w$ . Then  $\text{cl}(G')$  deformation retracts to  $\text{cl}(G)$ , so they have the same total Betti number. If  $H \leq G'$  then any induced copy of  $H$  in  $G'$  must contain  $v$  and  $w$ ; but then for  $G'' = G \sqcup v$  we get  $H \not\leq G''$ , and again the total Betti number of  $\text{cl}(G'')$  is one more than of  $\text{cl}(G)$ .  $\square$

## References

- [Ada14] Michał Adamaszek, *Extremal problems related to Betti numbers of flag complexes*, Discrete Appl. Math. **173** (2014), 8–15. MR 3202285
- [CS07] M. Chudnovsky and P. Seymour, *Excluding induced subgraphs*, Surveys in combinatorics 2007, London Math. Soc. Lecture Note Ser., vol. 346, Cambridge Univ. Press, Cambridge, 2007, pp. 99–119. MR 2252791
- [Chu14] Maria Chudnovsky, *The Erdős-Hajnal conjecture—a survey*, J. Graph Theory **75** (2014), no. 2, 178–190. MR 3150572
- [EH89] P. Erdős and A. Hajnal, *Ramsey-type theorems*, Discrete Appl. Math. **25** (1989), no. 1-2, 37–52, Combinatorics and complexity (Chicago, IL, 1987). MR 1031262 (90m:05091)
- [Fek23] M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten.*, Math. Z. **17** (1923), 228–249 (German).
- [JŚ03] Tadeusz Januszkiewicz and Jacek Świątkowski, *Hyperbolic Coxeter groups of large dimension*, Comment. Math. Helv. **78** (2003), no. 3, 555–583. MR 1998394 (2004h:20058)
- [KKTW01] D Král', J Kratochvíl, Z Tuza, and G. J. Woeginger, *Graph-theoretic concepts in computer science: 27th international workshop, wg 2001 boltenhagen, germany, june 14–16, 2001 proceedings*, ch. Complexity of Coloring Graphs without Forbidden Induced Subgraphs, pp. 254–262, Springer Berlin Heidelberg, Berlin, Heidelberg, 2001.
- [PS92] H. J. Prömel and A. Steger, *The asymptotic number of graphs not containing a fixed color-critical subgraph*, Combinatorica **12** (1992), no. 4, 463–473. MR 1194736 (93h:05093)



[vLW01] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, second ed., Cambridge University Press, Cambridge, 2001. MR 1871828 (2002i:05001)

## A $K_4$ -free graphs

**Preliminaries.** We keep the notational standards introduced in Section 2. We recall, that  $N[v] = N_G[v]$  denotes the closed neighborhood of vertex  $v$  in a graph  $G$ , that is, the set of neighbors of  $v$  together with  $v$ . We, however, modify the definition of the open neighborhood from Section 3. Throughout the appendix we assume that  $N(v) = N_G(v)$  is the subgraph of  $G$  induced by neighbors of  $v$ . That is, it is not only the set of neighbors as in Section 3. (For further considerations of the closed neighborhoods, it is not important whether we consider the subgraph or just the set of vertices. )

Since we plan to use Lemma 2.3 quite heavily, it pays of to set up certain additional notational conventions. Once we fix  $v$  and the order of the neighbors,  $v_1, \dots, v_d$  we define  $G^i = G - N[v_i] - \{v_1, \dots, v_{i-1}\}$  for  $i \in [d]$ . That is, the inequality in Lemma 2.3 can be rewritten as

$$\mathbf{b}(G) \leq \sum_{i=1}^d \mathbf{b}(G^i). \quad (8)$$

We also denote by  $k_i$  the size of the set  $V(N[v_i]) \cup \{v_1, \dots, v_{i-1}\}$ , that is  $G^i$  has  $n_i = n - k_i$  vertices.

**Lemma A.1.** *Let  $v_1, \dots, v_d$  be vertices forming a cut in  $G$  and let  $C$  be one of the components of  $G - \{v_1, \dots, v_d\}$  and  $G'$  be the union of the remaining components. Then*

$$\mathbf{b}(G) \leq \mathbf{b}(C)\mathbf{b}(G') + \sum_{i=1}^d \mathbf{b}(G^i).$$

The proof of this lemma is essentially the same as the proof of Lemma 2.3 in Adamaszek's paper [Ada14].

*Proof.* This lemma is obtained by an iterative application of Lemma 2.2. We remove all the vertices  $v_1, \dots, v_d$  one by one in the given order. Finally, we use that  $\mathbf{b}(G - \{v_1, \dots, v_d\}) = \mathbf{b}(C)\mathbf{b}(G')$  by Lemma 2.1.  $\square$

**The main bound.** We prove the following bound for  $K_4$ -free graphs.

**Theorem A.2.** *Let  $G$  be a graph with  $n$  vertices and without an induced copy of  $K_4$ . Then*

$$\mathbf{b}(G) \leq \Theta_2^n = 2^{n/3} \approx 1.2599^n.$$

*If, in addition,  $G$  contains a vertex of degree at most 3 which is not in a component consisting of a single triangle, then*

$$\mathbf{b}(G) \leq (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^n.$$

This first bound is asymptotically optimal as witnessed by the disjoint union of triangles.

Given that  $\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6} \approx 0.9618$ , the improvement from the second bound is very minor. However, it will be our crucial tool for ruling out 4-regular graphs.

**Minimal counterexample approach.** The proof is in principle given by induction in the spirit of previous proofs; however, some new ingredients are needed. From practical point of view, it is better to reformulate the induction in this case as the minimal counterexample approach. That is, we will assume that  $G$  is a counterexample to Theorem A.2 with the least number of vertices and we will gradually narrow the set of possible counterexamples until we show that such  $G$  cannot exist. It is easy to check that the theorem is valid for  $n = 1$  or  $n = 2$ .

## A.1 Roots of suitable polynomials

As our approach in previous sections suggest, we will need to know the roots of several suitable polynomials. Here we extend the considerations from Section 2. Given an ordered  $t$ -tuple of positive integers  $(a_1, \dots, a_t)$ , we will consider the equation

$$1 = x^{-a_1} + x^{-a_2} + \dots + x^{-a_t}. \quad (9)$$

This can be understood as a polynomial equation after multiplying with a suitable power of  $x$ . We are interested in a solution of this equation for  $x \in [1, \infty)$ . Note that the right hand side is at least 1 for  $x = 1$  and it is a decreasing function in  $x$  tending to 0. Therefore, there is a unique solution, which we denote by  $r_{a_1, \dots, a_t}$ . In our previous terminology,  $\Gamma_d = r_{d+1, \dots, 2d}$  and  $\Theta_d = r_{d+1, \dots, d+1}$  where there are  $d$  arguments. We will frequently use the following simple observation.

**Lemma A.3.** *Whenever  $\Omega$  is a real number such that  $\Omega \geq r_{a_1, \dots, a_t}$ , then  $\Omega^n \geq \Omega^{n-a_1} + \dots + \Omega^{n-a_t}$ , for any positive integer  $n$ .*

*Proof.* It is sufficient to prove  $1 \geq \Omega^{-a_1} + \dots + \Omega^{-a_t}$ . This immediately follows from the definition of  $r_{a_1, \dots, a_t}$ .  $\square$

We will need to know the approximative numerical values of  $r_{a_1, \dots, a_t}$  for various  $t$ -tuples  $(a_1, \dots, a_t)$ , so that we can mutually compare them. We present the values important for this section in Table 3; we also include some of the important values that we met previously.

We will also often use monotonicity, that is, if  $(b_1, \dots, b_t) \geq (a_1, \dots, a_t)$  entry-by-entry, then  $r_{b_1, \dots, b_t} \leq r_{a_1, \dots, a_t}$ . This allows us to skip computing precise values for many sequences  $(a_1, \dots, a_t)$ .

## A.2 Initial observations about the minimal counterexample

**Lemma A.4.** *Let  $G$  be a disconnected graph. Then  $G$  is not a minimal counterexample to Theorem A.2.*

$(a_1, \dots, a_t)$	the root	approx. value of the root	approx. value of $\Theta_2^{-a_1} + \dots + \Theta_2^{-a_t}$
(3, 3)	$\Theta_2 = 2^{1/3}$	1.2599	1
(6, 6, 6, 6)	$\Theta_2 = 2^{1/3}$	1.2599	1
(5, 7, 10, 10, 11, 11, 12, 12)	$r_{5,7,10,\dots,12}$	1.2590	
(6, 6, 9, 10, 11, 11, 12, 13)	$r_{6,6,9,\dots,13}$	1.2590	
(6, 6, 7, 8, 9)	$r_{6,6,7,8,9}$	1.2564	
(1, 7)	$r_{1,7}$	1.2554	
(5, 6, 6, 8)	$r_{5,6,6,8}$	1.2541	
(5, 6, 7, 7)	$r_{5,6,7,7}$	1.2519	
(4, 5, 6)	$\Gamma_3$	1.24985	0.9618
(5, 5, 5)	$3^{1/5}$	1.2457	0.9449
(3, 4)	$\Gamma_2$	1.2207	0.8969

Table 3: Solutions of Equation (9) for suitable  $t$ -tuples and values  $\Theta_2^{-a_1} + \dots + \Theta_2^{-a_t}$  for some of them.

*Proof.* The proof follows directly from Lemma 2.1. Indeed, let  $H_1, \dots, H_m$  be the components of  $G$ , where  $m \geq 2$ . Let  $n_i$  be the size of  $H_i$ . For contradiction, let us assume that  $G$  is a minimal counterexample to Theorem A.2. Then  $b(H_i) \leq \Theta_2^{n_i}$ . If in addition,  $H_i$  is not a triangle and it contains a vertex of degree at most 3, then  $b(H_i) \leq (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^{n_i}$ . Therefore, Lemma 2.1 gives  $b(G) \leq \Theta_2^{n_1+\dots+n_m}$  and, in addition,  $b(G) \leq (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^{n_1+\dots+n_m}$  if at least one  $H_i$  is not a triangle and it contains a vertex of degree at most 3. This contradicts that  $G$  is a counterexample to Theorem A.2.  $\square$

### A.3 Vertices of degree at most 2.

We begin by excluding vertices of degree at most 2.

**Lemma A.5.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then the minimum degree of  $G$  is at least 3.*

*Proof.* For contradiction assume the minimum degree of  $G$  is less than 3.

Trivially, the minimum degree of  $G$  cannot be zero (otherwise  $\mathbf{b}(G) = 0$ ).

If the minimum degree of  $G$  equals 1, let  $v$  be a vertex in  $G$  of degree 1. Let  $v_1$  be the neighbor of  $v$ . Lemma 2.3 gives

$$\mathbf{b}(G) \leq \mathbf{b}(G - N[v_1]).$$

This immediately gives that  $G - N[v_1]$  is a smaller counterexample.

It remains to consider the case when the minimum degree of  $G$  equals 2. Let  $v$  be a vertex of degree 2 and let  $v_1$  and  $v_2$  be its neighbors. If possible, we pick  $v$  so that  $v, v_1$  and  $v_2$  do not induce a component consisting of a single triangle. Lemma 2.3 gives

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2).$$

Note that the size of  $G^1$ , as well as of  $G^2$ , is at most  $n - 3$ .

If  $G$  is a minimal counterexample to Theorem A.2, then

$$\mathbf{b}(G) \leq \Theta_2^{n-3} + \Theta_2^{n-3} = \Theta_2^n$$

which gives the required contradiction for the first bound in Theorem A.2.

If, in addition,  $v, v_1$  and  $v_2$  do not induce a component consisting of a single triangle, then the size of  $G^1$  or of  $G^2$  is at most  $n - 4$ .

Since  $G$  is a minimal counterexample to Theorem A.2, we get

$$\mathbf{b}(G) \leq (\Theta_2^{n-3} + \Theta_2^{n-4}) = (\Theta_2^{-3} + \Theta_2^{-4})\Theta_2^n \leq (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^n$$

where the last inequality  $\Theta_2^{-3} + \Theta_2^{-4} < \Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}$  can be checked in Table 3. Therefore  $G$  is not a counterexample to Theorem A.2.  $\square$

## A.4 Vertices of degree 3

We continue our analysis by excluding vertices of degree 3.

**Proposition A.6.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then the minimum degree of  $G$  is at least 4.*

We need a number of lemmas ruling out various cases.

**Lemma A.7.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  does not contain a vertex  $v$  of degree 3 such that the open neighborhood  $N(v)$  consists of three isolated points.*

*Proof.* For contradiction, let us assume that  $G$  contains such a vertex  $v$  and let  $v_1, v_2$ , and  $v_3$  be its neighbors. As usual, Lemma 2.3 gives

$$b(G) \leq b(G^1) + b(G^2) + b(G^3).$$

We already know that the minimum degree of  $G$  is at least 3 by Lemma A.5. Since  $v_1, v_2$ , and  $v_3$  are three isolated points we get that the sizes of the three graphs on the right-hand side are at least  $n - 4, n - 5$  and  $n - 6$ .

If  $G$  is a minimal counterexample to Theorem A.2, by Lemma A.3 we get

$$b(G) \leq \Theta_2^{n-4} + \Theta_2^{n-5} + \Theta_2^{n-6} = (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^n.$$

This is the required contradiction. (Note that we have assumed only the weaker bound in Theorem A.2 for the graphs  $G^1, G^2$ , and  $G^3$ , but we still could derive the stronger bound for  $G$ .)  $\square$

By the previous lemma, we have ruled out a case when a vertex of degree three sees three isolated vertices. Now we will focus on the case when it sees an isolated vertex and an edge. At first we do not rule it out completely but set up some necessary condition.

**Lemma A.8.** *Let  $G$  be a minimal counterexample to Theorem A.2. If  $G$  contains a vertex  $v$  of degree 3 such that the open neighborhood  $N(v)$  consists of an edge and an isolated vertex, then all neighbors of  $v$  have degree 3.*

*Proof.* We know that the minimum degree of  $G$  is at least 3 by Lemma A.5. For contradiction, let us assume that  $G$  contains a vertex  $v$  such that it has three neighbors  $v_1, v_2$  and  $v_3$ ;  $\deg v_1 \geq 4$ ,  $\deg v_2, v_3 \geq 3$ , and the induced subgraph of  $G$  on  $\{v_1, v_2, v_3\}$  consists of an edge and an isolated vertex. Without loss of generality we assume that  $v_1$  and  $v_2$  are not connected with an edge (otherwise we swap  $v_2$  and  $v_3$ ). As usual, Lemma 2.3 gives

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3).$$

The size of all three graphs on the right-side is at most  $n - 5$ .

Therefore

$$\mathbf{b}(G) \leq 3\Theta_2^{n-5} \leq (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^n$$

since  $3\Theta_2^{-5} < \Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}$  which follows from Table 3 (or from the convexity of the function  $\Theta_2^x$ ).

□

Now we may rule out the case of a vertex of degree 3 which sees two edges in its neighborhood.

**Lemma A.9.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  does not contain a vertex  $v$  of degree 3 such that the open neighborhood  $N(v)$  consists of the path of length 2.*

*Proof.* For contradiction, let  $v$  be a vertex in  $G$  contradicting the statement of the lemma and let  $v_1, v_2$  and  $v_3$  be its neighbors. Without loss of generality,  $v_3$  is adjacent to  $v_1$  and to  $v_2$  but  $\{v_1, v_2\}$  is not an edge.

We need to distinguish some cases and subcases.

(i) First we assume that  $\deg v_3 = 3$ .

(a) Now we consider a subcase  $\deg v_2 = 3$ . See Figure 1.

In this subcase let  $w_2$  be the unique neighbor of  $v_2$  different from  $v$  and  $v_3$ . Let  $C$  be the edge  $vv_3$  and  $G' = G - \{v, v_1, v_2, v_3\}$ . Then Lemma A.1

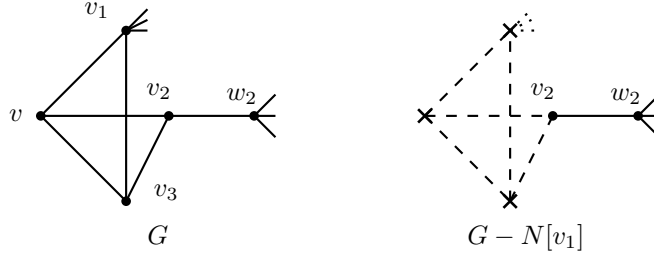


Figure 1: Subcase (ia),  $G$  and  $G - N[v_1]$ .

gives

$$\mathbf{b}(G) \leq \mathbf{b}(C)\mathbf{b}(G') + \mathbf{b}(G - N[v_1]) + \mathbf{b}(G - N[v_2] - v_1).$$

We observe that in the graph  $G - N[v_1]$ , the vertex  $v_2$  has degree 1. Thus we further get  $\mathbf{b}(G - N[v_1]) \leq \mathbf{b}(G - N[v_1] - N[w_2])$  by Lemma 2.3.

Note that  $\mathbf{b}(C) = 1$  and the size of  $G'$  is  $n-4$ . We also know that the size of  $G - N[v_2] - v_1$  is at most  $n-5$ . Finally, the size of  $G - N[v_1] - N[w_2]$  is at most  $n-6$ , even if  $w_2$  and  $v_1$  are neighbors. As usual, if  $G$  is a minimal counterexample to Theorem A.2, we get

$$\mathbf{b}(G) \leq 1 \cdot \Theta_2^{n-4} + \Theta_2^{n-6} + \Theta_2^{n-5}$$

as required.

- (b) If we consider a subcase  $\deg v_1 = 3$ , it can be solved analogously to the previous subcase by swapping  $v_1$  and  $v_2$ .
- (c) Finally, we consider the subcase  $\deg v_1 \geq 4$  and  $\deg v_2 \geq 4$ . Here we use the usual bound via Lemma 2.3 which gives

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3).$$

The size of  $G^1$  is at most  $n-5$ , the size of  $G^2$  is at most  $n-6$ , and the size of  $G^3$  is at most  $n-4$ . Therefore, we get a contradiction as above.

- (ii) Now we consider the case  $\deg v_3 \geq 4$ .



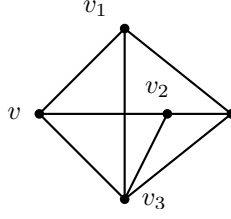


Figure 2: A graph occurring in case (ii).

If at least one of the vertices  $v_1$  or  $v_2$  has degree 4, or if  $\deg v_3 \geq 5$ , we use again the bound

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3).$$

The sizes of the three graphs on the right-hand side are either at least  $n - 5$  or they are at least  $n - 4$ ,  $n - 5$  and  $n - 6$  respectively. This yields the required contradiction eventually using that  $3\Theta_2^{-5} < \Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}$ .

Finally, we know that  $\deg v_1 = \deg v_2 = 3$  and  $\deg v_3 = 4$ . In such case either  $v_3$  and  $v_1$  have a single common neighbor (namely  $v$ ), or  $v_3$  and  $v_2$  have a single common neighbor (again  $v$ ), or we get the graph on Figure 2. (Indeed, if the rightmost vertex in Figure 2 has degree at least 4 then repeating the analysis above in (ii) for  $v_1$  instead of  $v$  gives the desired contradiction.) In the first case, we get a contradiction with Lemma A.8 for  $v_1$ . The second case is symmetric. In the last case, the independence complex of  $G$  consists of two edges and an isolated vertex; therefore  $\mathbf{b}(G) = 2 \leq (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^5$ . A contradiction.

□

Now we may rule out the only remaining case of minimum degree 3 when we have 3-regular graph where the open neighborhood  $N(v)$  of every vertex  $v$  consists of an edge and an isolated vertex.

**Lemma A.10.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  is not a cubic (3-regular) graph such that the open neighborhood  $N(v)$  of every vertex  $v$  consists*

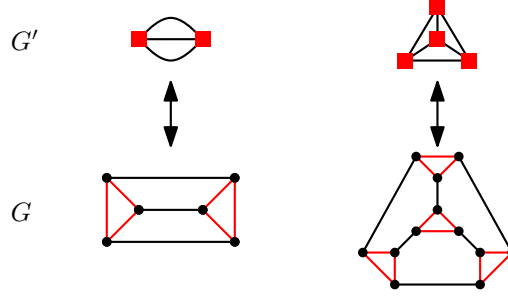


Figure 3: One to one correspondence between  $G$  and  $G'$ .

of an edge and isolated vertex.

*Proof.* For contradiction assume  $G$  is a minimal counterexample to Theorem A.2 and  $G$  satisfies the condition (C) that the open neighborhood  $N(v)$  of every vertex  $v$  consists of an edge and isolated vertex. Equivalently, the condition (C) can be reformulated so that  $G$  is a cubic graph where every vertex is incident to exactly one triangle. By contracting each triangle to a point, graphs satisfying (C) are in one to one correspondence with 3-regular multigraphs. (We allow multiple edges but we disallow loops.) See some examples on Figure 3. Let  $G'$  be the multigraph obtained from  $G$  by contracting the triangles of  $G$ . First, we show that  $G'$  is actually a graph. Indeed, if  $G'$  contains a triple edge, then  $G'$  must be the graph on the left part of Figure 3, as  $G$  is connected. In such case  $b(G) = 1$  since the independence complex of  $G$  is the 6-cycle. If  $G'$  contains a double edge, then  $G$  contains a subgraph as on Figure 4. The vertices  $a$  and  $v_1$  may or may not be neighbors. Let  $C$  be the subgraph of  $G$  induced by the vertices  $u, w, x, y, z$ . We get  $b(C) = 0$  since the independence complex of  $C$  is the path of length 4. Consequently, Lemma A.1 gives

$$b(G) \leq b(G - N[v_1]) + b(G - N[v_2] - v_1) \leq \Theta_2^{n-4} + \Theta_2^{n-5} < (\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})\Theta_2^n.$$

This yields the required contradiction.

Now we know that  $G'$  is a graph. We distinguish two cases: either  $G'$  contains an induced path of length 2 or not.

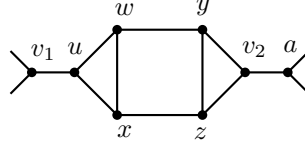


Figure 4: Part of  $G$  corresponding to a double-edge.

If  $G'$  does not contain an induced path of length 2, let us consider any vertex  $s$  of  $G'$ . We get that any pair of neighbors of  $s$  is adjacent. Therefore  $G'$  is the graph  $K_4$  (as  $G'$  is cubic and connected as well). We get that  $G$  is the graph on the right part of Figure 3 and it remains to bound  $b(G)$  for this particular graph. We need to show that  $b(G) \leq \Theta_2^{12}(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}) \approx 15.3893$ . Given that  $b(G)$  is an integer, our aim is in fact to show  $b(G) \leq 15$ .

We choose a vertex  $v$  of  $G$  arbitrarily, and we choose its neighbors  $v_1, v_2, v_3$  so that  $v_2$  and  $v_3$  are adjacent. We use the usual bound via Lemma 2.3 which gives

$$b(G) \leq b(G^1) + b(G^2) + b(G^3).$$

This bound would not be in general sufficient for a vertex  $v$  with such a neighborhood; however, we will show that three summands on the right hand-side are small enough integers for the graph at hand. The size of  $G^1$  is 8, the sizes of  $G^2$  and  $G^3$  are 7. Since  $G$  is a minimal counterexample, the first summand may be bounded by  $\Theta_2^8 \approx 6.3496$ . Given that this is an integer, we bound the first summand by 6. Similarly, we can bound the remaining two summands, but this time we use the stronger conclusion of Theorem A.2 which allows to bound each of the summands by  $\Theta_2^7(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}) \approx 4.8473$ , that is, we may bound these summands by 4. Altogether, we get  $b(G) \leq 14$  which gives the required contradiction.

Finally, it remains to consider the case when  $G'$  contains an induced path of length 2. In this case,  $G$  contains the subgraph on Figure 5. (Some of the pairs of vertices  $w_i$  and  $w_j$  may be adjacent.)

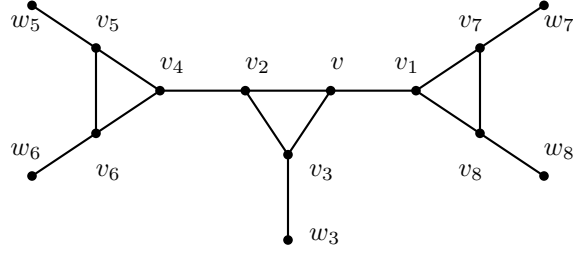


Figure 5: Part of  $G$  corresponding to an induced path of length 2.

We use the usual bound via Lemma 2.3 which gives

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3).$$

For the required contradiction, it would not be sufficient to check the orders of the graphs on the right hand-side. However, we may get a better bound for  $\mathbf{b}(G^1)$  by applying Lemma 2.3 again to this graph.

For the cut  $\{v_4, v_3\}$  in  $G^1$  we get

$$\mathbf{b}(G - N[v_1]) \leq \mathbf{b}(G - N[v_1] - N[v_4]) + \mathbf{b}(G - N[v_1] - N[v_3] - v_4).$$

The orders of the two graphs on the right hand-side of this inequality are  $n - 8$ . The orders of the graphs  $G - N[v_2] - v_1$  and  $G - N[v_3] - \{v_1, v_2\}$  are  $n - 5$ . Since  $G$  is a minimal counterexample, we get

$$\mathbf{b}(G) \leq \Theta_2^n(2\Theta_2^{-5} + 2\Theta_2^{-8}) = \Theta_2^n(3\Theta_2^{-5}) \leq \Theta_2^n(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6})$$

as required. (The equality in the middle follows since  $\Theta_2 = 2^{1/3}$ ; the last inequality follows from Table 3.) This gives the required contradiction.  $\square$

Now we conclude everything to a proof of Proposition A.6.

*Proof of Proposition A.6.* Let  $G$  be a minimal counterexample to Theorem A.2. By Lemma A.5 we know that the minimum degree of  $G$  is at least 3. It is, therefore, sufficient to show that  $G$  does not contain a vertex of degree 3.

Since  $G$  is  $K_4$ -free the open neighborhood of any vertex must be triangle-free. That is, the open neighborhood of any vertex are either three isolated points; an edge and a point; or a path of length 2. Any of these options is ruled out by Lemmas A.7; A.9; and A.8 and A.10, respectively.  $\square$

## A.5 Vertices of degree at least 6

Now we bound the maximum degree of a possible minimal counterexample.

**Lemma A.11.** *Let  $G$  a minimal counterexample to Theorem A.2, then the degree of every vertex of  $G$  is at most 5.*

*Proof.* For contradiction, let  $v$  be a vertex of degree  $d \geq 6$  in  $G$ . By Lemma 2.2, we have

$$\mathbf{b}(G) \leq \mathbf{b}(G - v) + \mathbf{b}(G - N[v]).$$

Since  $G$  is a minimal counterexample to Theorem A.2, we get that the right hand side of the inequality above is at most  $\Theta_2^{n-1} + \Theta_2^{n-(d+1)} \leq \Theta_2^{n-1} + \Theta_2^{n-7}$ . Since  $r_{1,7} < \Theta_2$  (see Table 3), we get  $\mathbf{b}(G) \leq \Theta_2^{n-1} + \Theta_2^{n-7} \leq \Theta_2^n$ . Together with Proposition A.6, this contradicts the fact that  $G$  is a counterexample to Theorem A.2.  $\square$

## A.6 Vertices of degree 4

We continue our analysis by excluding vertices of degree 4. As above, we let  $G$  to be a minimal counterexample on  $n$  vertices. By Proposition A.6 we know that the minimum degree of  $G$  is at least 4, and by Lemma A.11, the maximum degree of  $G$  is at most 5. These are already quite restrictive conditions. On the other hand, the treatment of vertices of degree 4 is perhaps the most complicated part of the proof of Theorem A.2.

We consider a vertex  $v$  of degree 4 (if it exists). We check its open neighborhood  $N(v)$ , and depending on  $N(v)$  and on the degrees of vertices of  $N(v)$  in  $G$ , we rule out many cases how may  $N(v)$  look like. Once we rule out these cases, we get graphs with certain structure; and this structure helps us to estimate  $\mathbf{b}(G)$  more precisely. This will rule out the remaining cases.

Let  $v_1, \dots, v_4$  denote the vertices of  $N(v)$ , and recall the bound (8)

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3) + \mathbf{b}(G^4),$$

where  $G^i$  stands for  $G - N[v_i] - \{v_1, \dots, v_{i-1}\}$ . We will often alternate the order of the vertices  $v_1, \dots, v_4$  in order to get the best bound.

We also recall that  $k_i$  is set up so that  $G^i$  has  $n - k_i$  vertices. If we show that  $r_{k_1, \dots, k_4} \leq \Theta_2$ , then we are done, since we obtain

$$\mathbf{b}(G) \leq \Theta_2^n$$

by Lemma A.3. This is the required contradiction. In particular, we achieve this task, if  $(k_1, \dots, k_4) \geq (6, 6, 6, 6)$  or  $(k_1, \dots, k_4) \geq (5, 6, 7, 7)$ , up to possibly permuting  $k_1, \dots, k_4$ ; see Table 3. (This is not the same as permuting  $v_1, \dots, v_4$ ; permuting the vertices may yield an essentially different values of  $k_1, \dots, k_4$ .) On the other hand, it is insufficient to achieve that  $(k_1, \dots, k_4)$  is  $(5, 5, 7, 7)$  or  $(5, 6, 6, 7)$ , since  $r_{k_1, \dots, k_4} > \Theta_2$  in these cases (very tightly). This will complicate our analysis.

Now, let us inspect the possible neighborhoods  $N(v)$ . Since  $G$  is  $K_4$ -free, we get that  $N(v)$  is triangle-free. There are 7 options for the isomorphism class of  $N(v)$  depicted on Figure 6.

The discussion above immediately gives that the last two options cannot occur for a minimal counterexample.

**Lemma A.12.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  does not contain vertex  $v$  such that  $N(v)$  is isomorphic to  $I_4$  or  $P_2 + I_2$ ; see Figure 6.*

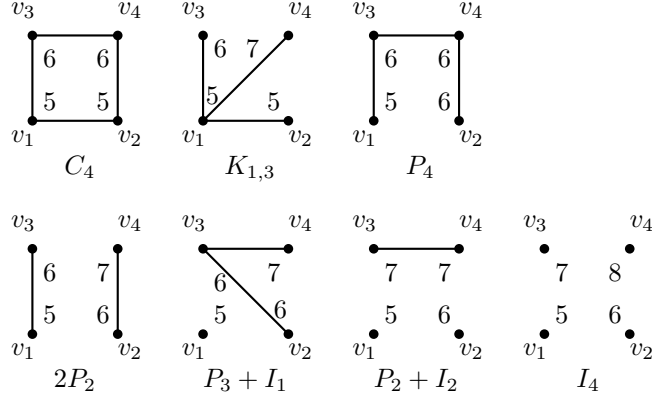


Figure 6: The possible isomorphism classes of  $N(v)$ . The number at a vertex  $v_i$  denotes  $k_i$  under the condition that the degree of  $v_i$  in  $G$  is 4.

*Proof.* For contradiction, there is such  $v$  in the minimal counterexample. We already know that we may assume that the minimum degree of  $G$  is at least 4. Therefore, from the discussion above we get  $(k_1, \dots, k_4) \geq (5, 6, 7, 7)$ , which yields the required contradiction.  $\square$

Our next task is to show that if  $G$  is a minimal counterexample which contains a vertex of degree 4, then  $G$  is actually 4-regular. We do this in two steps. First we significantly restrict the possible isomorphism classes of  $N(v)$  where  $v$  is a vertex of degree 4 incident to a vertex of degree 5. Next, we analyze the remaining options in more details so that we may rule them out as well.

**Lemma A.13.** *Let  $G$  be a minimal counterexample to Theorem A.2. Let  $v$  be a vertex of degree 4 in  $G$ , which is incident to a vertex of degree greater or equal to 5. Then one of the following options hold.*

- (a)  $N(v)$  is isomorphic to  $C_4$ , one vertex of  $N(v)$  has degree 5 in  $G$  and the three remaining vertices have degrees 4 in  $G$ .
- (b)  $N(v)$  is isomorphic to  $C_4$ , two opposite vertices of  $N(v)$  have degrees 5 in  $G$  and the two remaining vertices have degrees 4 in  $G$ .

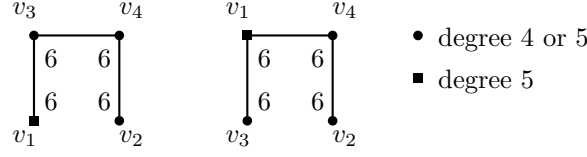


Figure 7: Excluding the case that  $N(v)$  is isomorphic to  $P_4$ . The label at  $v_i$  denotes  $k_i$  under the condition that all bullet vertices have degree 4. (In general, it is a lower bound for  $k_i$ .)

(c)  $N(v)$  is isomorphic to  $K_{1,3}$ , one vertex of  $N(v)$  has degree 5 in  $G$  and the three remaining vertices have degrees 4 in  $G$ .

*Proof.* Let  $v$  be the vertex from the statement. We gradually exclude all remaining cases. By Lemma A.12, we already know that  $N(v)$  is not isomorphic to  $P_2 + I_2$  or  $I_4$ .

First let us consider the case that  $N(v)$  is isomorphic to  $2P_2$  or  $P_3 + I_1$ . Let us choose  $v_1, \dots, v_4$  according to Figure 6. Since one of the vertices  $v_1, \dots, v_4$  has degree 5, we get that  $(k_1, \dots, k_4) \geq (6, 6, 6, 7)$  or  $(k_1, \dots, k_4) \geq (5, 6, 7, 7)$  or  $(k_1, \dots, k_4) \geq (5, 6, 6, 8)$ . Therefore this option is excluded since the three roots  $r_{6,6,6,6}$ ,  $r_{5,6,7,7}$ , and  $r_{5,6,6,8}$  are less than  $\Theta_2$ ; see Table 3.

Now let us consider the case that  $N(v)$  is isomorphic to  $P_4$ . At least one vertex of  $N(v)$  has degree 5 in  $G$ . Up to isomorphism, there are two (non-exclusive) options depicted at Figure 7. Depending on these options we label the vertices of  $N(v)$  by  $v_1, \dots, v_4$  according to Figure 7. In both cases, we get  $(k_1, \dots, k_4) \geq (6, 6, 6, 6)$  which contradicts that  $G$  is a minimal counterexample.

Let us continue with the case that  $N(v)$  is isomorphic to  $K_{1,3}$ . If there is only one vertex in  $N(v)$  of degree 5 in  $G$ , we get the case (c) of the statement of this lemma. Therefore, we may assume that  $N(v)$  contains at least two vertices of degree 5 in  $G$  and we want to exclude this case. We label the vertices of  $N(v)$  by  $v_1, \dots, v_4$  according to Figure 6. Up to a self-isomorphism of  $K_{1,3}$  we may assume that  $v_2$  has degree 5 and also  $v_1$  or  $v_3$  has degree 5. Therefore, we get  $(k_1, \dots, k_4) \geq (6, 6, 6, 7)$  or  $(k_1, \dots, k_4) \geq (5, 6, 7, 7)$  which gives the required



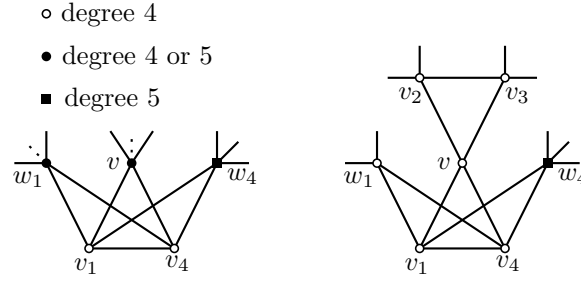


Figure 8: Forbidden subgraph.

contradiction.

Finally, it remains to consider the case that  $N(v)$  is isomorphic to  $C_4$ . In this case, it is sufficient to exclude the case that  $N(v)$  contains two vertices of degree 5 in  $G$  which are neighbors. If there are two such vertices, we label the vertices of  $N(v)$  by  $v_1, \dots, v_4$  so that  $v_1$  and  $v_2$  have degrees 5. Then  $(k_1, \dots, k_4) \geq (6, \dots, 6)$  which is the required contradiction.  $\square$

**Lemma A.14.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  does not contain the graph on 5 vertices from Figure 8, left, as an induced subgraph, where  $\deg(v_1) = \deg(v_4) = 4$ ,  $\deg(w_4) = 5$  and  $\deg(w_1), \deg(v) \in \{4, 5\}$ .*

*Proof.* Assume by contradiction  $G$  contains such a subgraph, and observe that the neighborhood  $N(v_1)$  is isomorphic to  $K_{1,3}$ . Therefore, Lemma A.13 gives that  $\deg(v) = \deg(w_1) = 4$ .

Now, let us focus on  $N(v)$ . Since we know all neighbors of  $v_1$  and  $v_4$ , we get that  $N(v)$  is isomorphic to one of the graphs  $2P_2$  or  $P_2 + I_2$ . But Lemma A.12 excludes the latter case. In addition, Lemma A.13 implies that all neighbors of  $v$  have degree 4 in  $G$ . Let  $v_2$  and  $v_3$  be the two remaining neighbors of  $v$ ; see Figure 8, right.

Now, we want to use Lemma 2.3 on  $v$ . In this case  $(k_1, \dots, k_4) = (5, 6, 6, 7)$  which is not sufficient but we may gain a slight improvement if we inspect the

graphs on the right hand-side of Lemma 2.3 in this case:

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3) + \mathbf{b}(G^4). \quad (10)$$

We have that the size of  $G^i$  is  $n - k_i$ . However, we may also check that  $G^2 = G - N[v_2] - v_1$  contains a vertex of degree at most 3 which is not contained in a component of  $G^2$  consisting of a single triangle. Indeed,  $v_4$  is such a vertex. (Note that  $w_1$  or  $w_4$  may or may not belong to  $G^2$ ). Similarly,  $G^3$  contains a vertex of degree at most 3 which is not contained in a component of  $G^3$  consisting of a single triangle, which is again witnessed by  $v_4$ .

Therefore, since  $G$  is a minimal counterexample to Theorem A.2, we get

$$\mathbf{b}(G^2), \mathbf{b}(G^3) \leq \Theta_2^{n-6}(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}).$$

Hence (10) gives

$$\mathbf{b}(G) \leq \Theta_2^n(\Theta_2^{-5} + (\Theta_2^{-6} + \Theta_2^{-6})(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}) + \Theta_2^{-7}).$$

We get a contradiction to the assumption that  $G$  is a counterexample to Theorem A.2 as  $r_{5,7,10,10,11,11,12,12} \leq \Theta_2$ ; see Table 3.

□

Now we have enough tools to exclude the remaining cases of Lemma A.13.

**Lemma A.15.** *Let  $G$  be a minimal counterexample to Theorem A.2. If  $G$  contains a vertex of degree 4, then  $G$  is 4-regular.*

*Proof.* We know that  $G$  is connected by Lemma A.4, and has minimal degree 4 by Proposition A.6. For contradiction, let us suppose that  $G$  contains a vertex of degree 4 but  $G$  is not 4-regular. In particular,  $G$  contains a vertex  $v$  of degree 4 which is incident to a vertex of degree 5; thus one of the three options (a,b,c) in Lemma A.13 must hold.

First, let us consider the case (c), that is,  $N(v)$  is isomorphic to  $K_{1,3}$  and  $v$  is

incident to exactly one vertex of degree 5. Let us label the vertices of  $N(v)$  as in Figure 6. Now, there are two subcases, either  $v_1$  is the vertex of degree 5, or, without loss of generality,  $v_2$  is the vertex of degree 5.

In the first subcase,  $v_1$  has a single neighbor  $w$  different from  $v, v_2, v_3$  and  $v_4$ . Now let us describe  $N(v_2)$ . By Lemma A.13,  $N(v_2)$  is isomorphic to  $C_4$  or  $K_{1,3}$  ( $v_1$  is a neighbor of  $v_2$  of degree 5). In addition,  $v$  and  $v_1$  belong to  $V(N(v_2))$ . We also have  $\deg_{N(v_2)}(v) = 1$  since  $\deg_G(v) = 4$  and  $v_2$  and  $v_3$  are not neighbors of  $v_2$ . Similarly, we deduce  $\deg_{N(v_2)}(v_1) \leq 2$ . This rules out both options,  $C_4$  and  $K_{1,3}$  for the isomorphism class of  $N(v_2)$ . A contradiction.

In the second subcase, we suppose that  $v_2$  has degree 5 in  $G$ . Therefore,  $v_1$  has degree 4 in  $G$  and consequently  $N(v_1)$  is isomorphic to  $K_{1,3}$ . Therefore, up to relabeling of the vertices,  $G$  contains the induced subgraph from Lemma A.14. This gives the required contradiction.

This way, we have ruled out the case (c) of Lemma A.13. Therefore, we may assume that any vertex  $v$  of degree 4 of  $G$ , incident to a vertex of degree 5, falls into the case (a) or (b) of Lemma A.13.

Now, let us consider an arbitrary vertex  $w$  of degree 5, incident to a vertex  $v$  of degree 4. Our aim is to show that  $N(w)$  is isomorphic either to  $C_5$ , or to  $C_4 + I_1$ ; see Figure 9.

By inspecting  $N(v)$ , we get that  $v$  and  $w$  have two common neighbors, say  $v_1$  and  $v_2$ , which are not incident. Now, we analogously inspect the 4-vertex graphs  $N(v_1)$ ,  $N(v_2)$  and so on (for the other vertices of degree 4 incident to  $w$ ), and we arrive at one of the two cases in Figure 9 (keeping the degree notation of Figure 8).

Therefore, it is sufficient to distinguish two subcases according to the isomorphism type of  $N(w)$ .

First we suppose that  $N(w)$  is isomorphic to  $C_5$ . Then all vertices of  $N(w)$  have degree 4 in  $G$ . Let us label the vertices of  $N(w)$  according to Figure 9 and let  $x$  be the neighbor of  $v$  different from  $w, v_1$  and  $v_2$ . By checking  $N(v)$  again,

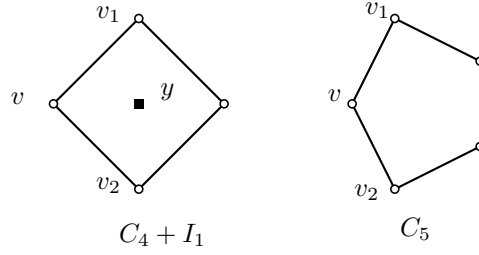


Figure 9: Possible isomorphism types of  $N(w)$ .

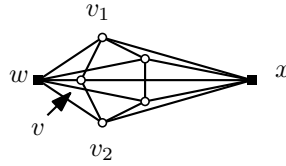


Figure 10: The resulting  $G$  if  $N(w)$  is isomorphic to  $C_5$ .

we see that  $v_1$  and  $v_2$  are neighbors of  $x$  as well. Then by checking  $N(v_1)$  and  $N(v_2)$  we get that all vertices of  $N(w)$  are incident to  $x$ , and we get that  $G$  is the graph on Figure 10. In this case, we easily observe that the independence complex of  $G$  consists of an edge and a cycle. Therefore  $\mathbf{b}(G) = 2 \leq \Theta_2^7$  which is the required contradiction.

Now, we suppose that  $N(w)$  is isomorphic to  $C_4 + I_1$ . Let us again label the vertices of  $N(w)$  according to Figure 9. In this case, the four vertices of  $C_4$  have degree 4 in  $G$ . (The last vertex  $y$  has degree 5 in  $G$ , but we do not need this information.) Analogously to the previous case, we deduce that there is another vertex  $x$  incident to the vertices of  $C_4$  in  $N(w)$ . The degree of  $x$  in  $G$  may be 4 or 5. See Figure 11. Let us apply Lemma A.1 to the cut formed by the vertices  $w$  and  $x$  (in this order, which is relevant for the lemma). We obtain

$$\mathbf{b}(G) \leq 1 \cdot \Theta_2^{n-6} + \Theta_2^{n-6} + \Theta_2^{n-6} = \frac{3}{4} \Theta_2^n.$$

As usual, this contradicts that  $G$  is a counterexample to Theorem A.2.  $\square$

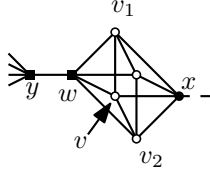


Figure 11: The resulting  $G$  if  $N(w)$  is isomorphic to  $C_4 + I_1$ .

**4-regular graphs.** Now we know that if a minimal counterexample  $G$  contains a vertex of degree 4, then it must be 4-regular. Our next step is to rule out this case.

We will often need to check that a certain graph satisfies the stronger condition in the statement of Theorem A.2. Here is a useful sufficient condition which allows us to avoid distinguishing various special cases.

**Lemma A.16.** *Let  $G$  be a connected 4-regular graph and let  $H$  be a proper subgraph of  $G$  such that the number of vertices of  $H$  is not divisible by 3. Then  $H$  contains a vertex of degree at most 3 in  $H$  which is not in a component consisting of a single triangle.*

*Proof.* Let us consider a component  $C$  of  $H$  which has the number of vertices not divisible by 3. In particular  $C$  is not a triangle. Since  $H$  is a proper subgraph of a connected 4-regular graph,  $C$  must contain a vertex of degree at most 3.  $\square$

In Lemma A.12 we have ruled out certain options for the neighborhood of a vertex of degree 4. Now, we may rule out further options.

**Lemma A.17.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  does not contain vertex  $v$  such that  $N(v)$  is isomorphic to  $2P_2$  or  $P_3 + I_1$ ; see Figure 6.*

*Proof.* By Lemma A.15, we know that  $G$  is 4-regular. For contradiction, let us assume that there is a vertex  $v$  such that  $N(v)$  is isomorphic to  $2P_2$  or  $P_3 + I_1$ . Let us label the neighbors of  $v$  according to Figure 6. In our usual notation, this gives  $(k_1, k_2, k_3, k_4) = (5, 6, 6, 7)$ . This is insufficient to rule out these cases

directly, but it will help us to focus on the stronger conclusion of Theorem A.2. Lemma 2.3 gives

$$\mathbf{b}(G) \leq \mathbf{b}(G^1) + \mathbf{b}(G^2) + \mathbf{b}(G^3) + \mathbf{b}(G^4)$$

where  $G^i = G - N[v_i] - \{v_1, \dots, v_{i-1}\}$  as usual. Now, let us consider two cases depending on whether the number of vertices of  $G$  is divisible by 3. If it is divisible by 3, Lemma A.16, together with the fact that  $G$  is a minimal counterexample, gives

$$\mathbf{b}(G) \leq \Theta_2^{n-5}(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}) + \Theta_2^{n-6} + \Theta_2^{n-6} + \Theta_2^{n-7}(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}).$$

If the number of vertices of  $G$  is not divisible by 3, we analogously get

$$\mathbf{b}(G) \leq \Theta_2^{n-5} + 2\Theta_2^{n-6}(\Theta_2^{-4} + \Theta_2^{-5} + \Theta_2^{-6}) + \Theta_2^{n-7}.$$

In both cases, we get the required contradiction, since  $r_{6,6,9,10,11,11,12,13} \leq \Theta_2$  as well as  $r_{5,7,10,10,11,11,12,12} \leq \Theta_2$ . See Table 3.  $\square$

Now, we may also rule out an open neighborhood isomorphic to  $K_{1,3}$ .

**Lemma A.18.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  does not contain a vertex  $v$  such that  $N(v)$  is isomorphic to  $K_{1,3}$ ; see Figure 6.*

*Proof.* For contradiction, there is such a vertex  $v$ . Let us label the neighbors of  $v$  according to Figure 6. By Lemma A.15, we know that  $G$  is 4-regular. Therefore, the only common neighbor of  $v_2$  and  $v_1$  is  $v$ . Similarly, the only common neighbor of  $v_2$  and  $v$  is  $v_1$ . Therefore  $N(v_2)$  must be isomorphic to  $2P_2$  or to  $P_2 + I_2$ . However this is already ruled out by Lemmas A.12 and A.17.  $\square$

Now let us establish two graph classes that will help us to work with 4-regular graphs such that the open neighborhood of every vertex is isomorphic either to the cycle  $C_4$  or to the path on 4 vertices  $P_4$ . The *triangular path* on  $n$

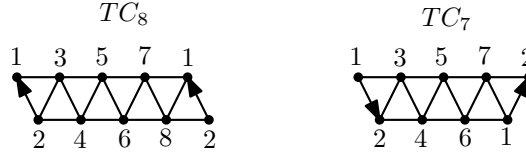


Figure 12: The graphs  $TC_8$  and  $TC_7$  (after the identification of the vertices labeled 1 and 2).

vertices is the graph  $TP_n$  such that  $V(TP_n) := [n]$  and

$$E(TP_n) := \left\{ ij \in \binom{[n]}{2} : |i - j| \leq 2 \right\}.$$

Similarly, we define *triangular cycle* so that we consider the distance cyclically. That is, we get a graph  $TC_n$  such that  $V(TC_n) := V(TP_n) = [n]$  and

$$E(TC_n) := \left\{ ij \in \binom{[n]}{2} : i - j \pmod{n} \in \{n - 2, n - 1, 1, 2\} \right\}.$$

See Figure 12.

If we consider the clique complex  $\text{cl}(TC_n)$ , then we get a triangulation of an annulus for  $n \geq 8$  even, whereas we get a triangulation of the Möbius band for  $n \geq 7$  odd. We establish the following structural result for graphs with the remaining two options for open neighborhoods.

**Lemma A.19.** *Let  $G$  be a connected 4-regular graph such that the open neighborhood of every vertex is isomorphic either to  $C_4$  or to  $P_4$ . Then  $G$  is isomorphic to  $TC_n$  for some  $n \geq 6$ .*

*Proof.* Let us consider the clique complex  $\text{cl}(G)$ . By the condition on the neighborhoods, we get that  $\text{cl}(G)$  is a triangulated surface, possibly with boundary.

Let  $k$  be the number of vertices of  $G$  such that their neighborhood is isomorphic to  $C_4$  and  $\ell$  be the number of remaining vertices. Therefore, by double counting, we get that  $\text{cl}(G)$  has  $k + \ell$  vertices,  $2(k + \ell)$  edges, and  $\frac{4}{3}k + \ell$  triangles. That is, the Euler characteristic  $\chi(\text{cl}(G))$  equals  $k + \ell - 2(k + \ell) + \frac{4}{3}k + \ell = \frac{1}{3}k$ .

However; surfaces with nonnegative Euler characteristic are rare, which will help us to rule out many options.

It is easy to check that  $G$  has at least 6 vertices because the closed neighborhood of a single vertex has already 5 vertices.

If  $\ell = 0$ , then  $k \geq 6$  and therefore  $\chi(\text{cl}(G)) \geq 2$ . This leaves an only option that  $\text{cl}(G)$  is a sphere,  $\chi(\text{cl}(G)) = 2$  and  $k = 6$ . Consequently (by checking how to extend a neighborhood of arbitrary vertex), we get that  $G = K_{2,2,2} = TC_6$ .

If  $\ell > 0$ , then  $\text{cl}(G)$  must be a surface with boundary and the only options are the disc (with Euler characteristic 1), the annulus and the Möbius band (the latter two have Euler characteristic 0).

In the case of a disc, we get  $k = 3$ . We say, that a vertex  $v$  of  $G$  is a  $C_4$ -vertex, if  $N(v)$  is isomorphic to  $C_4$ . The interior vertices of the disc are precisely the  $c_4$ -vertices. By a local check of the neighborhoods, we see that every  $C_4$  vertex is adjacent to at least two  $C_4$ -vertices, and therefore, the  $C_4$ -vertices form a triangle in  $G$ . We now count the edges according to the number of  $C_4$ -vertices they contain. Note that no edge of the disc connects two boundary vertices, as such edge  $e$  would separate the disc into two regions, and in the region with no interior vertex there would be a boundary vertex, not in  $e$ , of degree at most 2; a contradiction. Thus, each boundary vertex has exactly two neighbors in the boundary and two in the interior. The number of boundary edges is clearly  $l$ . To summarize, the total number of edges is  $3 + l + 2l$ , but it is also  $2(3 + l)$ , thus  $l = 3$ . This is a contradiction as then the triangle on the 3 boundary vertices is in  $\text{cl}(G)$ , eliminating the boundary of the disc.

Finally, it remains to consider the case of the annulus and the Möbius band. In this case,  $k = 3\chi(\text{cl}(G)) = 0$ ; so all the vertices are on the boundary of  $\text{cl}(G)$ . Now a simple local inspection gives that  $G$  is isomorphic to  $TC_n$  for  $n \geq 7$ . (We consider an arbitrary vertex  $v$  and its neighborhood  $N(v)$ , then we check the neighborhoods of the vertices of  $N(v)$  which locally determines the graph uniquely. We continue this inspection, until we reach a vertex from 'two direc-



tions'.)

□

Now we need to bound  $\mathbf{b}(TC_n)$  for  $n \geq 6$  in order to finish the case of graphs of minimum degree 4. First, we provide a bound for  $\mathbf{b}(TP_n)$  which will be useful for bounding  $\mathbf{b}(TC_n)$ .

**Lemma A.20.** *For  $n \neq 3$ , we have  $\mathbf{b}(TP_n) \leq 2^{n/4}$ . Furthermore  $\mathbf{b}(TP_3) = 2$ .*

*Proof.* It is easy to determine the first few initial values by checking the corresponding independence complexes. We obtain  $\mathbf{b}(TP_0) = 1$ ,  $\mathbf{b}(TP_1) = 0$ ,  $\mathbf{b}(TP_2) = 1$ ,  $\mathbf{b}(TP_3) = 2$ , and  $\mathbf{b}(TP_4) = 2$ , where  $TP_0$  stands for the empty graph.

Next, we use Lemma 2.3 to vertex  $n$  and its neighbors  $n - 1$  and  $n - 2$ . We get

$$\mathbf{b}(TP_n) \leq \mathbf{b}(TP_{n-4}) + \mathbf{b}(TP_{n-5}). \quad (11)$$

This further gives  $\mathbf{b}(TP_5) \leq 1$ ,  $\mathbf{b}(TP_6) \leq 1$ ,  $\mathbf{b}(TP_7) \leq 3$ , and  $\mathbf{b}(TP_8) \leq 4$ . Therefore,  $\mathbf{b}(TP_n) \leq 2^{n/4}$  for  $n \in [8] \setminus \{3\}$ . (Note that  $2^{7/4} \approx 3.3636$ .) Furthermore, it is trivial to show that  $\mathbf{b}(TP_n) \leq 2^{n/4}$  for  $n \geq 9$  by induction using (11). □

Now we bound  $\mathbf{b}(TC_n)$ .

**Lemma A.21.** *For  $n \geq 9$  we have  $\mathbf{b}(TC_n) \leq 2^{n/4}(2^{-1/2} + 2^{-1/4}) \approx 1.5480 \cdot 2^{n/4}$ .*

*Proof.* First remove the vertex  $n$  and then the vertex  $n - 1$  from  $TC_n$ . Lemma 2.2 then gives

$$\begin{aligned} \mathbf{b}(TC_n) &\leq \mathbf{b}(TC_n - n) + \mathbf{b}(TC_n - N[n]) \\ &\leq \mathbf{b}(TC_n - \{n, n - 1\}) + \mathbf{b}(TC_n - n - N[n - 1]) + \mathbf{b}(TC_n - N[n]) \\ &\leq \mathbf{b}(TP_{n-2}) + 2\mathbf{b}(TP_{n-5}). \end{aligned}$$

Therefore, Lemma A.20 gives

$$\mathbf{b}(TC_n) \leq 2^{(n-2)/4} + 2 \cdot 2^{(n-5)/4} = 2^{n/4}(2^{-1/2} + 2^{-1/4}).$$

□

Now we may rule out 4-regular graphs.

**Lemma A.22.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  is not a 4-regular graph.*

*Proof.* For contradiction, let us assume that there is such  $G$ . By Lemma A.4 we know that  $G$  is connected. By Lemmas A.12, A.17 and A.18 we know that the open neighborhood of every vertex in  $G$  is isomorphic either to  $C_4$  or to  $P_4$ . Lemma A.19 implies that  $G$  is isomorphic to  $TC_n$  for  $n \geq 6$ . Therefore, in order to obtain a contradiction, it is sufficient to show that  $b(TC_n) \leq \Theta_2^n = 2^{n/3}$ .

We treat separately the cases  $n \in \{6, 7, 8\}$ . The independence complex of  $TC_6$  consists of three edges and therefore  $b(TC_6) = 2$ . The independence complex of  $TC_7$  is the cycle  $C_7$  which gives  $b(TC_7) = 1$ . Finally, the independence complex of  $TC_8$  is a connected 3-regular graph (triangle-free) with 8 vertices, thus with 12 edges. Therefore  $b(TC_8) = 5$ . In all three cases, we easily see that  $b(TC_n) \leq 2^{n/3}$ .

Now we consider  $n \geq 9$ . Lemma A.21 gives  $b(TC_n) \leq 2^{n/4}(2^{-1/2} + 2^{-1/4})$ . Therefore, we need to check the inequality  $2^{-1/2} + 2^{-1/4} \leq 2^{n/12}$ . This inequality holds for  $n \geq 8$  since  $2^{8/12} \approx 1.5874$  while  $2^{-1/2} + 2^{-1/4} \approx 1.5480$ .

□

Proposition A.6 and Lemmas A.11, A.15 and A.22 together imply the following corollary.

**Proposition A.23.** *Let  $G$  be a minimal counterexample to Theorem A.2. Then  $G$  is a 5-regular graph.*

□

## A.7 5-regular graphs

It remains to rule out 5-regular graphs. We use an analogous approach as in the case of 4-regular graphs. Given a minimal counterexample  $G$ , which is 5-regular

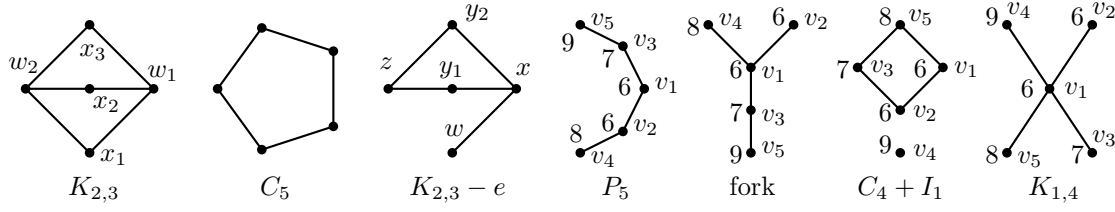


Figure 13: Triangle-free graphs with 5 vertices and at least 4 edges.

by Proposition A.23, and a vertex  $v$  of  $G$ , we consider all possible isomorphism classes of  $N(v)$ . Those are triangle free graphs on 5 vertices. All triangle free graphs on 5 vertices with at least 4 edges are depicted on Figure 13. All other triangle-free graphs on 5 vertices are subgraphs of  $P_5$  or  $K_{1,4}$ . (It is easy to check both claims from the well known list of graphs on 5 vertices.)

We use the standard approach via Lemma 2.3 to rule out the cases when  $N(v)$  does not have many edges.

**Lemma A.24.** *Let  $G$  be a minimal counterexample to Theorem A.2 and let  $v$  be any vertex of  $G$ . Then  $N(v)$  contains at least 5 edges.*

*Proof.* For contradiction,  $G$  is a minimal counterexample and  $v$  is a vertex of  $G$  such that  $N(v)$  contains at most 4 edges. We know that  $G$  is 5-regular. Therefore,  $N(v)$  is one of the four graphs with 4 edges on Figure 13, or their subgraph. Let us label the vertices of  $N(v)$  according to Figure 13 (we fix one choice of a subgraph if  $G$  has less than 4 edges). We use Lemma 2.3 to  $v$ . In our standard notation, we get  $(k_1, \dots, k_5) \geq (6, 6, 7, 8, 9)$  (if  $G = C_4 + I_1$ , we have to permute last two coordinates). Therefore,  $G$  cannot be a counterexample since  $r_{6,6,7,8,9} < \Theta_2$ ; see Table 3.  $\square$

Therefore, it remains to consider the connected 5-regular graphs such that the open neighborhood of every vertex is isomorphic to  $K_{2,3}$ ,  $C_5$  or  $K_{2,3} - e$ ; see Figure 13. Fortunately, such graphs are very rare. In fact, we will show that there are only two such graphs. One of them is the graph of the icosahedron, which we denote by  $G_{\text{ico}}$ . The second graph is the join (as a graph, not as a

simplicial complex) of  $C_5$  and  $I_3$  which we denote by  $C_5 \star I_3$ . That is,  $C_5 \star I_3$  is the graph with the vertex set  $V(C_5 \star I_3) = V(C_5) \cup V(I_3)$ , assuming that  $V(C_5)$  and  $V(I_3)$  are disjoint, and with the set of edges

$$E(C_5 \star I_3) = E(C_5) \cup \{uv : u \in C_5, v \in I_3\}.$$

**Lemma A.25.** *Let  $G$  be a 5-regular graph such that the open neighborhood of every vertex is isomorphic to  $K_{2,3}$ ,  $C_5$  or  $K_{2,3} - e$ . Then  $G$  is isomorphic to  $G_{\text{ico}}$  or to  $C_5 \star I_3$ .*

*Proof.* Let us first consider the case that  $G$  contains a vertex  $v$  such that  $N(v)$  is isomorphic to  $K_{2,3} - e$ . Let us label the vertices of  $N(v)$  according to Figure 13. Now, let us focus on  $N(w)$ . It contains  $x$  and  $v$ , which are neighbors. Moreover,  $\deg_{N(w)} v = 1$  since  $v$  has degree 5 in  $G$  and  $v$  is also incident to  $y_1, y_2$  and  $z$  which are not incident with  $w$ . Therefore  $N(w)$  must be isomorphic to  $K_{2,3} - e$  as well, which implies that  $\deg_{N(w)} x = 3$ . But this is a contradiction, since  $x$  has too many neighbors, namely  $v, w, y_1, y_2$  and two other neighbors which are incident to  $w$ . Altogether,  $G$  cannot contain a vertex such that its open neighborhood is isomorphic to  $K_{2,3} - e$ .

Now, let us consider the case that  $G$  contains a vertex  $v$  such that  $N(v)$  is isomorphic to  $K_{2,3}$ . Let us label the vertices of  $N(v)$  according to Figure 13. Now let us focus on  $N(w_1)$ . It contains a subgraph formed by the vertices  $v, x_1, x_2$  and  $x_3$  isomorphic to  $K_{1,3}$ . Therefore  $N(w_1)$  cannot be isomorphic to  $C_5$ , so it must be isomorphic to  $K_{2,3}$ . Now we focus on  $N(x_1)$ . By checking  $N(v)$ , we get that  $\deg_{N(x_1)} v = 2$ . By an analogous argument,  $\deg_{N(x_1)} w_1 = 2$  since we already know that  $N(w_1)$  is isomorphic to  $K_{2,3}$ , and  $w_1 v$  is an edge in  $N(x_1)$ . Therefore  $N(x_1)$  cannot be isomorphic to  $K_{2,3}$  which implies that it is isomorphic to  $C_5$ . Analogously, we deduce that  $N(x_2)$  and  $N(x_3)$  are isomorphic to  $C_5$ .

As  $G$  is a connected 5-regular graph, in order to show that  $G$  is isomorphic to  $C_5 \star I_3$ , it is sufficient to show that  $N(x_1) = N(x_2) = N(x_3)$ . We will show  $N(x_1) = N(x_2)$  and the other equality  $N(x_1) = N(x_3)$  will be analogous. Let us again focus on  $N(v)$ . The two edges  $vw_1$  and  $vw_2$  belong simultaneously

to  $N(x_1)$  and  $N(x_2)$ . Now, if we refocus to  $N(w_1)$ , we see that the two edges of  $N(x_1)$  incident with  $w_1$  belong also to  $N(x_2)$ , as  $w_1$  has degree 5 in  $G$ . By repeating this argument for  $w_2$ , we get that  $N(x_1)$  and  $N(x_2)$  share a 4-path on 5 vertices. As both  $N(x_1)$  and  $N(x_2)$  are isomorphic to  $C_5$  we conclude  $N(x_1) = N(x_2)$ .

Finally, it remains to consider the case that the open neighborhood of every vertex of  $G$  is isomorphic to  $C_5$ . In this case, the clique complex  $\text{cl}(G)$  is a closed triangulated surface without boundary. Let  $n$  be the number of vertices of  $G$ . By double-counting,  $\text{cl}(G)$  contains  $\frac{5}{2}n$  edges and  $\frac{5}{3}n$  triangles. Therefore, the Euler characteristic  $\chi(\text{cl}(G))$  equals  $n - \frac{5}{2}n + \frac{5}{3}n = \frac{n}{6}$ . In particular,  $\chi(\text{cl}(G))$  is positive; therefore  $\text{cl}(G)$  must be the sphere or the projective plane. The case of projective plane cannot occur, because in such case, we would have  $n = 6\chi(\text{cl}(G)) = 6$ , forcing  $G = K_6$  in the unique 6-vertex triangulation of the projective plane; but then  $\text{cl}(G)$  is the 5-simplex, a contradiction. Hence we know that  $\text{cl}(G)$  is the sphere and  $n = 6\chi(\text{cl}(G)) = 12$ . However, it is well known that the only 5-regular graph that triangulates the sphere is the graph of the icosahedron.  $\square$

It remains to rule out the two cases from the previous lemma as minimal counterexamples.

**Lemma A.26.** *Neither  $C_5 \star I_3$  nor  $G_{\text{ico}}$  is a counterexample to Theorem A.2.*

*Proof.* It is easy to compute that  $\text{b}(C_5 \star I_3) = 2$  since the corresponding independence complex is the union of a 5-cycle and a triangle. Since  $2 < 2^{8/3} = \Theta_2^8$ , we get that  $C_5 \star I_3$  is not a counterexample to Theorem A.2.

It is a bit harder to determine  $\text{b}(G_{\text{ico}})$  precisely since the corresponding independence complex is 2-dimensional. Let  $K$  be the independence complex of  $G_{\text{ico}}$ . Let  $v$  be an arbitrary vertex of  $G_{\text{ico}}$ . Then  $v$  is not incident to a subgraph of  $G_{\text{ico}}$  forming the wheel graph  $W_5$ ; see Figure 14. This gives that the link  $\text{lk}_K v$  is the independence complex of  $W_5$ , that is, the disjoint union  $C_5 + I_1$  of  $C_5$  and a vertex. Let  $K'$  be the complex obtained from  $K$  by removing all edges which are

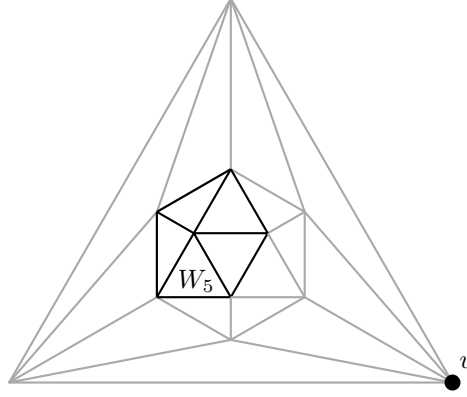


Figure 14: A vertex  $v$  of the icosahedron and the graph formed by the non-neighbors of  $v$ .

not incident to any triangle. This means removing 6 edges. Then the link of every vertex of  $K'$  is isomorphic to  $C_5$ . Now, the same reasoning as in the last part of the proof of Lemma A.25 gives that  $K'$  is the boundary of the icosahedron (but the vertices are significantly permuted when compared to the icosahedron for  $G_{\text{ico}}$ ). This implies that  $K$  is homotopy equivalent to the wedge of one 2-sphere and six 1-spheres. We obtain  $b(G_{\text{ico}}) = 7$ . Given that  $7 < 8 < 2^{12/3} = \Theta_2^{12}$ , we get that  $G_{\text{ico}}$  is not a counterexample to Theorem A.2.  $\square$

Now we conclude everything and obtain the final result.

*Proof of Theorem A.2.* For contradiction, there is a minimal counterexample  $G$  to Theorem A.2. By Proposition A.23,  $G$  is 5-regular. By Lemmas A.24 and A.25,  $G$  must be isomorphic to  $G_{\text{ico}}$  or to  $C_5 \star I_3$ . However, Lemma A.26 excludes these two options.  $\square$